Due Dates: One question of your choice (either #1 or #2) is due by **11:59PM Sunday, 25 September** and will be graded on completion only (1 point). Your full answers to all questions are due by **11:59PM Wednesday, 28 September**. Each solution will be graded for correctness and clarity (4 points per question). Read the course information page for further details.

At the top of your write-up for each problem, estimate the amount of time you spent on the problem, list your collaborators, and briefly describe the nature of your collaboration.

1. Some complexity arguments

   (a) Define $f : \mathbb{N} \to \mathbb{R}^+$ by $f(n) = 3n^2 + 1000n + 5000$. Show that $f(n) = O(n^2)$

   (b) Suppose $g : \mathbb{N} \to \mathbb{R}^+$ satisfies $g(n) = \Omega(n^2)$. Show that $g(n) = \Omega(n)$.

   (c) Suppose $f, g : \mathbb{N} \to \mathbb{R}^+$ satisfy $f(n) = O(n)$ and $g(n) = O(n)$. Let $h(m, n) = f(m)+g(n)$. Show that $h(m, n) = O(m + n)$.

   (d) Suppose $f, g : \mathbb{N} \to \mathbb{R}^+$ and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ is defined by $h(m, n) = f(m) + g(n)$. If $h(m, n) = O(m + n)$ is it necessarily true that $f(n) = O(n)$ and $g(n) = O(n)$? If so, prove it. If not, give a counterexample.

   For the two questions involving $O(m + n)$, you can use this definition.

   Definition 1 *Let* $f, g : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$*. Then* $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{O}(\mathbf{g}(\mathbf{x}, \mathbf{y}))$ *if* $\exists C > 0, N > 0$ *such that* $x \geq N$ *and* $y \geq N \implies f(x, y) \leq Cg(x, y)$.

2. Let $S = \{s_1, s_2, ..., s_n\}$ be a set of students seeking partners for an upcoming class project. Suppose further that each s_k has created a preference list, ranking the other $n - 1$ students. Assume that n is even.

   (a) Provide a definition for *stable matching* in this context. ~~(Note that among other things, your definition should handle the case where n is odd.)~~

   (b) Is a given set of students and preference lists guaranteed to have a stable matching? (Here's a little hint: no.) Give a (small, I hope) example to answer this question.

   ~~(c) Devise an algorithm for this matching problem. It should always terminate, it should either generate a stable matching if one exists or terminate with an empty matching otherwise, and it should be as efficient as possible.~~

   ~~(d) Prove your algorithm correct and analyze its efficiency.~~

   (c) Rewrite the Gale-Shapley algorithm from page 6 of Kleinberg & Tardos in a way that makes it make sense within the context of our one-set-of-students context. Change GS as little as you can.

   (d) Show by example that your modified algorithm can result in an unstable matching.

   (e) Show exactly where the proof of GS (i.e. the proof of claim 1.6 on pages 8-9 plus the proofs that lead up to it) fails for your modified algorithm.

Follow-up questions

As usual, I'll just put some more things to think about down here in case you have extra time, brain-space, and inclination to think about it. The grader and I won't grade anything you write about these questions, but I'm always delighted to talk with you about these and other ideas.

Anyway, here are some things you could think about. Problem 2 above concerns one of the many variations on stable matching problems. How might you adjust Gale-Shapley to deal with the following variations? What does *stable* mean in each case? How would you need to modify the algorithm? Would our original G-S proof hold up, and if not, where would it break down? Which variations correspond to what real-life applications? etc.

- There's only one set of people/items to be matched. (This is Problem 2 above.)
- The two sets to be matched have different sizes.
- *Indifference* is allowed in people's preference lists (i.e., ties are allowed–"I like Alice best, then it's a tie between Bob and Cecilia, and Doug comes last")
- People don't have to list everyone in the other set in their preference lists.
- People in M can match to multiple people in W. (This is the situation of the CS Match, where $M = \{students\}$ and $W = \{courses\}$.)
- ...?

Question 2 adapted from an assignment by Layla Oesper