Math 5707 Stable Matchings

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> *"It is a truth universally acknowledged, that a single man in possession of a good fortune must be in want of a wife."*

Recall 1. A binary relation $\leq$ is a *linear* or *total order* if it is antisymmetric ($a \leq b$ and $b \leq a$ implies $a = b$), transitive, and total ($a \leq b$ or $b \leq a$).

Definition 2. Let $G = (A, B, E)$ be a bipartite graph. For each $v \in A \sqcup B$, let $\leq_v$ be a linear order on $N(v)$. Call the collection $\{\leq_v\}_{v \in V}$ a *set of preferences* for G. A matching M of G is *stable* if for every edge $e \in E \setminus M$, there exists an edge $f \in M$ such that $e = vx$, $f = vy$, and $x <_v y$.

Example 3. National Resident Matching Program

Definition 4 (Stable matching algorithm)**.** Let $G = (A, B, E)$ be a bipartite graph and $\{\leq_v\}_{v \in V}$ a set of preferences.

(a) For each vertex $a \in A$, let $b \in N(a)$ be the $\leq_a$-maximal vertex, and add ab to M.
(b) For each vertex $b \in B$ incident to multiple edges in M, let $a \in N(b)$ be the $\leq_b$-maximal vertex such that $ab \in M$, and delete *from* E (and thus also from M) all edges in M incident to b except for ab.
(c) Repeat the steps above until unmatched $a \in A$ are all isolated.

> *"And now nothing remains for me but to assure you in the most animated language of the violence of my affection."*

Theorem 5 (2.1.4, Gale–Shapley 1962)**.** *Given a bipartite graph $G = (A, B, E)$ and a set of preferences, the stable matching algorithm produces a stable matching M. Moreover, if $G = K_{n,n}$ is a complete bipartite graph, then M is perfect.*

Proof. Note that if $b \in B$ is in M after some round, it will always be in M. Furthermore, b will only "trade up" and be matched with increasingly more desirable vertices. Therefore we will never get the same M again (except immediately, when we terminate). As there are only finitely many possible M, the process will terminate at some point.

Suppose $ab \in E(G) \setminus M$. If a never proposed to b, that means a is currently matched with someone $b' \in B$ more preferable ($b <_a b'$). Otherwise, a proposed to b at some point. Then ab was deleted from M at some point, which could only happen if b had a better match $a' \in A$ available. Since b only trades up, b is currently matched with someone $a'' \in A$ more preferable. □

Exercise 6. Stable matchings might not exist in non-bipartite graphs. For example, a triangle where each vertex prefers its right neighbour.

Example 7. Stable matchings are not (necessarily) unique. For example, a 4-cycle where each vertex prefers its neighbour on the right. This also means whether A or B propose will lead to different results in the algorithm.