Math 5707 Exam 3

There are 6 problems, each worth 7 points. **Turn in solutions for (at most) 5 of them.** If you turn in work for all 6 problems, an arbitrary subset of 5 problems will be graded. Be sure to justify all your work: answers without sufficient justification will receive no credit.

Grading. The first two major errors cost 3 and 2 points, respectively. Minor errors cost 1 point each. Incoherent solutions are awarded a 0 without regard to the aforementioned scheme. A subset of some common (major, unless otherwise labelled) errors are listed after each problem.

Problem 1. Let $G = (V, E)$ be a graph on n vertices and $\overline{G}$ its complement (see Diestel §1.1). Prove the following inequalities involving the chromatic numbers $\chi(G)$ and $\chi(\overline{G})$.

(a) $\chi(G) \cdot \chi(\overline{G}) \geq n$.
(b) $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$.

[**Hint:** Relate $\chi(\overline{G})$ with $\alpha(G)$, the size of the largest independent set. Use (a) for (b), even if you didn't manage to prove (a).]

Solution. Fix a colouring c of G with $a := \chi(G)$ colours and a colouring c' of $\overline{G}$ with $b := \chi(\overline{G})$ colours. On one hand, $S := \{(c(v), c'(v)) : v \in V\}$ is a subset of $[a] \times [b]$, and so

$$|S| \leq |[a] \times [b]| = ab.$$

On the other hand, for each pair of distinct vertices $u, v \in V(G) = V(\overline{G})$, note that $uv \in E(G) \sqcup E(\overline{G})$, so u and v must be coloured differently in c or c', and hence

$$|S| = n,$$

proving (i).

Taking the square root of

$$(a + b)^2 = a^2 + 2ab + b^2 = (a^2 - 2ab + b^2) + 4ab = (a - b)^2 + 4ab \geq 4ab$$

gives (ii). $\square$

Grading. Failure to do one of the parts (correctly).

Problem 2. For $n \in \mathbb{N}$, prove that there is a tournament on n vertices with at least $n!2^{1-n}$ directed Hamilton paths.

Solution. Obtain a tournament T from K^n by choosing a direction for each edge independently and randomly with probability $1/2$ for either direction. Let X be the random variable denoting the number of directed Hamilton paths. For each permutation $\sigma = (v_1, \ldots, v_n)$ of the n vertices, let X_σ be the indicator that $v_1 v_2 \ldots v_n$ is a directed Hamilton path. For this to happen, the $n-1$ edges must be directed correctly. The probability of this happening is 2^{1-n}, as the edge directions are independent. Therefore, the expectation is $\mathbb{E}[X_\sigma] = 2^{1-n}$. As $X = \sum_\sigma X_\sigma$, where σ runs over all $n!$ permutations of the vertex set, by linearity of expectation, we have

$$\mathbb{E}[X] = \sum_\sigma \mathbb{E}[X_\sigma] = n!2^{1-n}.$$

Since expectation is a (weighted) average, there is *some* orientation with at least $\mathbb{E}[X] = n!2^{1-n}$ directed Hamilton paths, as desired. □

Grading. Using probability without (at least informally) defining a probability space, not using independence.

Problem 3. There are n gnomes operating the gnomeship UMN *Enterprise* MATH-5707. This ship has $k \leq n$ different stations, each of which must be occupied by a gnome who is trained for that station. It is desired that **any** subset of k gnomes can operate the ship (in the event that an arbitrary subset of $n-k$ gnomes are ~~killed~~ sleeping). One way to do this is to train all n gnomes for all k stations. However, as training is costly, the Gnome Resources Allocation Provisioning Hegemony asks you to minimize the total number of gnomes trained for the stations. To avoid being executed, you should propose a training scheme and prove that it is optimal, in the sense that it is impossible to have fewer number of total trainings. [**Hint:** The G.R.A.P.H. understands a graph theoretic proof if and only if its relation to the present situation is made explicit.]

Solution. Let $G = \{g_1, \ldots, g_n\}$ be the set of n gnomes and $S = \{s_1, \ldots, s_k\}$ the set of k stations. Create a bipartite graph (S, G, E) where a gnome $g \in G$ is joined to a station $s \in S$ if and only if g is trained for s. Note that a subset $H \subseteq G$ of k gnomes can operate the ship if and only if there is a complete matching of S that matches each station to a gnome in H. We wish to minimize the number of edges $|E|$ in such a graph subject to this constraint.

Let E consists of a matching $\{s_i g_i : 1 \leq i \leq k\}$ and all edges between S and g_i for $i > k$. Let $H \subseteq G$ be a set of $|H| = k$ gnomes. Match each $g_i \in H$ with $i \leq k$ to s_i. Since the remaining gnomes are trained for all possible stations, surely a matching exists. This shows that $|E| = k + (n-k)k$ works.

Conversely, suppose that (S, G, E) has fewer than $k + (n-k)k = k(n-k+1)$ edges. Not all k stations can have degree at least $n-k+1$, so some station s_i has degree at most $n-k$. This means s_i has at least $n-(n-k) = k$ non-neighbours in G. Pick a subset H of k non-neighbours of s_i in G to show that no matching can cover s_i and H. □

Grading. Incorrect training scheme, incorrect proof that a training scheme works, failure to establish a (correct) lower bound on the number of trainings required.

Problem 4. For $n \geq 3$, prove that in any edge colouring of a K^n with 2 colours, there is a Hamilton cycle that is the union of two monochromatic paths. [**Hint:** A path is monochromatic if all its edges are coloured equally. A monochromatic Hamilton cycle is the union of two monochromatic paths in many different ways.]

Solution. Say a Hamilton cycle is *k-part* if it is a union of k *independent* monochromatic paths. We first claim that it suffices to find a 3-part Hamilton cycle. Indeed, suppose a Hamilton cycle is a union of three independent monochromatic paths P_1, P_2, P_3. By the pigeonhole principle, using two colours (holes) for three paths (pigeons), at least two paths, say P_1 and P_2, are coloured equally. As P_1 and P_2 necessarily intersect, $P_1 \cup P_2$ is a monochromatic path, which, together with P_3, shows that the Hamilton cycle is the union of two monochromatic paths.

Apply induction on n. The base case $n = 3$ is immediate, as the three edges are independent monochromatic paths (of length one) that form a 3-part Hamilton cycle.

Now let $G = K^{n+1}$ and colour the edges with colours red and blue. Fix a vertex $v \in V(G)$ and consider $G - v$. By induction, $G - v$ has a 2-part Hamilton cycle $v_0 v_1 \dots v_n$, where $v_0 = v_n$. WLOG, assume $v_0 v_1 \dots v_k$ is (monochromatically) red, and $v_k v_{k+1} \dots v_n$ is blue (or contains no edges), for some $1 \leq k \leq n$. Consider the colour of the edge $vv_0 = vv_n$. If vv_n is blue, then vv_1 is monochromatic, $v_1 v_2 \dots v_k$ is red, and $v_k \dots v_n v$ is blue. This means $vv_1 \dots v_n v$ is a 3-part Hamilton cycle. Otherwise, vv_0 is red. Set $t = \min\{k, n-1\}$. Then $vv_0 v_1 \dots v_t$ is (monochromatically) red, $v_t v_{t+1} \dots v_{n-1}$ is blue, and $v_{n-1} v$ is a monochromatic path of length 1. This means $vv_0 v_1 \dots v_{n-1} v$ is a 3-part Hamilton cycle. □

Grading. Not considering the case of a monochromatic Hamilton cycle in the inductive step, or (minor) claiming it is easy without proof; considering too many cases (minor).

Problem 5. Formulate and prove the generalization of Ford–Fulkerson (Theorem 6.2.2) for networks with multiple source and sink vertices. [**Hint:** In a network, we assumed that a source vertex has only outgoing edges and a sink vertex has only incoming edges. You may wish to redefine or clarify terms such as *flow* (and its *total value*) and *cut* (and its *capacity*). **Do not reinvent the wheel.**]

Solution. Let $D = (V, E)$ be a digraph and (D, S, T, c) be a *generalized network*, where S and T are disjoint subsets of V consisting of source and sink vertices. As before, a source (resp. sink) vertex has no incoming (resp. outgoing) edges. A *flow* f of N is a function $E \to \mathbb{R}$ such that f is conserved at every vertex $x \in V \backslash (S \cup T)$, and feasible at every edge $e \in E$. A *cut* is a pair (X, Y) of subsets partitioning V such that $S \subseteq X$ and $T \subseteq Y$, and $c(X, Y)$ is its *capacity*, as before. Define the *total value* of a flow f, denoted $|f|$, by $f(S, V)$.

Generalized Ford–Fulkerson states: In every generalized network, the maximum total value of a flow equals the minimum capacity of a cut.

Let $N = (D, S, T, c)$ be a generalized network. Define an ordinary network $N' = (D', s', t', c')$ as follows: Start with N. Add a new vertex s' to $V' := V(D')$ and, for each $s \in S$, add an edge (s', s) with capacity $c(s, V)$. Similarly, add a new vertex t' to V' and, for each $t \in T$, add an edge (t, t') with capacity $c(V, t)$.

Given a flow f of N, define a flow f' of N' by setting the values on the new edges in the only possible way to make f' conserved on the vertices of $S \sqcup T$. This clearly creates a bijection between flows of N and flows of N'. Furthermore, corresponding flows have the same total value:

$$|f| = f(S, V) = f'(\{s'\} \cup S, V') = |f'| \,.$$

In particular, the maximum total value of a flow of N is equal to the maximum total value of a flow of N', which, by the ordinary Ford–Fulkerson Theorem, is equal to the minimum capacity of a cut of N'.

It remains to show that this minimum capacity is the same for N as well. Take a cut (X, Y) of N, and consider the cut $(\{s'\} \cup X, Y \cup \{t'\})$ of N'. As $S \subseteq X$ and $T \subseteq Y$, the new edges involving s' or t' do not appear in the sum $c(X, Y)$. The summands are exactly the same as for $c'(\{s'\} \cup X, Y \cup \{t'\})$, and hence their capacities are equal. This means

$$\min c(X, Y) \geq \min c'(X', Y'), \tag{*}$$

where (X, Y) runs over all cuts of N and (X', Y') runs over all cuts of N'.

Finally, take a cut (X', Y') of N' with minimum capacity. Let $X = X' \backslash \{s'\}$ and $Y = Y' \backslash \{t'\}$. Note that (X, Y) may not be a cut if X' and Y' do not contain S and T, respectively. Suppose $s \in S \backslash X'$. Then

$$c'(X', Y') = c'(X', Y' \backslash \{s\}) + c'(X', s).$$

Note that $c'(X', s) = c'(s', s) = c(s, V) = c'(s, V') \geq c'(s, Y \backslash \{s\})$, so

$$c'(X', Y') \geq c'(X', Y' \backslash \{s\}) + c'(s, Y \backslash \{s\}) = c'(X' \cup \{s\}, Y' \backslash \{s\}).$$

As $c'(X', Y')$ was minimum over all cuts, the above is actually an equality. Therefore, we may assume $X' \supseteq S$, and analogously assume $Y' \supseteq T$. Then (X, Y) is in fact a cut of N, and therefore we have equality in (*), as desired. □

Grading. Not formulating the generalization.

Problem 6. An orientation is **acyclic** if it contains no (directed) cycles. Let Q_G denote the number of acyclic orientations of a graph G.

(a) Prove that $Q_G = Q_{G-e} + Q_{G/e}$ for each edge $e \in E(G)$.
(b) Prove that $Q_G = |P_G(-1)|$, where $P_G(k)$ is the chromatic polynomial defined in Exercise 5.18 of Diestel.

[**Hint:** In fact, $Q_G = (-1)^{|G|} P_G(-1)$.]

Solution. Let G be a graph and fix $e = \{x, y\} \in E(G)$. Given an acyclic orientation of G, by deleting the directed edge corresponding to e, no cycles would be created, and hence we obtain an acyclic orientation of $G - e$.

Conversely, let D be an acyclic orientation of $G - e$. For $i = \{0, 1, 2\}$, say D is *i-extendable* if exactly i of the orientations $D + (x, y)$ and $D + (y, x)$ of G are acyclic, and let q_i denote the number of i-extendable acyclic orientations of $G - e$. By definition we have

$$Q_G = q_1 + 2q_2$$

and

$$Q_{G-e} = q_0 + q_1 + q_2.$$

The acyclic orientation D is not 0-extendable. Indeed, if $D + (x, y)$ is not acyclic, then there is a directed cycle C. As D is acyclic, C must contain the directed edge (x, y). Then $C - (x, y)$ is a directed y–x path P_1 in D. Similarly, if $D + (y, x)$ is not acyclic, then there is a directed x–y path P_2 in D. Let z be the first vertex after y on yP_1x that is also on xP_2y, which exists because they share another vertex x. By definition yP_1z and zP_2y are independent, so yP_1zP_2y is a directed cycle in D, a contradiction. This means

$$q_0 = 0.$$

Suppose D is 2-extendable. This means there are no directed x–y or y–x paths in D. In particular, each common neighbour of x and y either dominates both x and y or is dominated by both x and y. As such, D descends[1] naturally to an orientation D' of G/e by identifying x and y.

If D' contains a cycle C, as D is acyclic, C must pass through the vertex corresponding to x and y, but then it exhibits the existence of a directed x–y or y–x path in D, a contradiction, so D' is acyclic. This process has a natural inverse[2], assigning an orientation D of $G - e$ to a given acyclic orientation D' of G/e. Furthermore, the act of "uncontracting" cannot possibly create a new directed cycle, so D is acyclic. This means

$$Q_{G/e} = q_2.$$

Putting these four displayed equations together yields (a).

From (the solutions to) Exercise 5.18 of Diestel, we know that the chromatic polynomial satisfies

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

We apply induction on $\|G\|$ to prove

$$Q_G = (-1)^{|G|} P_G(-1).$$

For the base case, there are no edges, we have $P_G(k) = k^{|G|}$ and so $Q_G = (-1)^{|G|} P_G(-1) = 1$. By part (a) and the induction hypothesis, we have

$$\begin{aligned} Q_G = Q_{G-e} + Q_{G/e} &= (-1)^{|G|} P_{G-e}(-1) + (-1)^{|G|-1} P_{G/e}(-1) \\ &= (-1)^{|G|} \big(P_{G-e}(-1) - P_{G/e}(-1) \big) \\ &= (-1)^{|G|} P_G(-1), \end{aligned}$$

as desired. □

Grading. Failure to do one of the parts (correctly).

[1] More explicitly (don't actually do this, I got bored writing the solutions so I'm just having some fun), let $p : V(G - e) \to V(G/e)$ be the map that sends x and y to v_{xy}, the vertex e contracts to, and is the identity otherwise. As an edge is an unordered pair, p induces $\tilde{p} : E(G-e) \to E(G/e)$, given by $\{u, v\} \mapsto \{p(u), p(v)\}$. An orientation of a graph (V, E) is in obvious bijection with a function $f : E \to V$ with $f(e) \in e$, recording the initial vertex, say, for each directed edge. By the discussion above, if f represents a 2-extendable orientation D, then there exists a unique map $\tilde{f}$ such that the following diagram commutes:

$$\begin{tikzcd} E(G-e) \arrow[r, "f"] \arrow[d, "\tilde{p}"'] & V(G-e) \arrow[d, "p"] \\ E(G/e) \arrow[r, "\tilde{f}"] & V(G/e) \end{tikzcd}$$

[2]Indeed, given $\tilde{f}$, there is a unique map f that makes the above diagram commutes.

Problem	Mean	Stdev	Mode
Problem 1 (7 points)	6.00	1.33	7
Problem 2 (7 points)	5.50	1.46	7
Problem 3 (7 points)	4.92	2.72	7
Problem 4 (7 points)	5.06	2.29	7
Problem 5 (7 points)	5.00	2.58	7
Problem 6 (7 points)	5.00	2.68	7
$\sum$ (35 points total)	23.94	7.96	

Counting multiplicities of $\{0, 1, \ldots, 7\}$ for all problems and students, the most common score for a problem is 7 by far, followed by 4 and 6, in that order.

Here are the letter grades and their corresponding ranges of total course score (out of 100).

Score Range	Letter Grade	Multiplicity
85–	A	4
75–84	$A-$	4
65–74	$B+$	1
55–64	B	3
45–54	$B-$	1
35–44	$C+$	4
25–34	C	1