Math 5707 Exam 1

Due at the beginning of lecture on Wednesday, March 5, 2014.

Please staple this sheet to the front of your solutions.

There are 6 problems, each worth 7 points. **Turn in solutions for (at most) 5 of them.** If you turn in work for all 6 problems, an arbitrary subset of 5 problems will be graded. Be sure to justify all your work: answers without sufficient justification will receive no credit.

You may use resources such as books and the Internet. Do not collaborate or consult human sources besides the instructor. Clearly indicate any outside sources consulted, and make sure to understand the solutions sufficiently to *explain them in your own words.* Solutions which the instructor views as insignificant alterations of outside sources will receive no credit.

Name: ______________________________

ID: ______________________________

Problem 1 (7 points)	
Problem 2 (7 points)	
Problem 3 (7 points)	
Problem 4 (7 points)	
Problem 5 (7 points)	
Problem 6 (7 points)	
$\sum$ (35 points total)	

Problem 1. Once upon a time, there was a village with 23 gnomes. Every gnome gave hats to 5 other gnomes. Is it possible that every gnome received hats from the same 5 gnomes to whom he gave hats?

Problem 2. Let $G = (V, E)$ be a graph on $n = |V|$ vertices. Suppose that $G - v$ is a tree for every vertex $v \in V$.

(i) How many edges does G have?
(ii) Determine the structure of G.

Problem 3. Devise an algorithm to perform the following task. Given a graph $G = (V, E)$, find a subset $S \subseteq V$ of vertices such that the induced subgraph $G[S]$ contains no edges, and that

$$|S| \geq \frac{|V|}{\Delta(G) + 1},$$

where $\Delta(G)$ denotes the maximum degree of the graph G.

Problem 4. Let $n \in \mathbb{N}$ be positive. For each pair of integers x and y such that $1 \leq x \leq y \leq n$, take a card and label one side with x and the other with y.

(i) How many cards are there?

Put these cards on top of each other to form a *deck*, such that sides touching each other have equal labels, *i.e.*, the back of a card has the same label as the front of the next card. You are allowed to flip over the cards when assembling such a deck. Note that a deck of cards have two numbers showing: the front of the first card and the back of the last card. We say that a deck is *orderly* if these two numbers are also equal.

(For example, write $[x, y]$ for a card with x on the front and y on the back. If $n = 3$, the following represents an orderly deck: $[1, 2], [2, 2], [2, 3], [3, 3], [3, 1], [1, 1]$. Note that $[3, 1]$ is the card $[1, 3]$ flipped over.)

(ii) Prove that it is possible to assemble an *orderly* deck using every card if and only if n is odd.

[**Hint:** Relate this to Euler tours of some graph.]

Problem 5. (Exercise 2.9 in Diestel.) Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n \in \mathbb{N}$. Show that there are (pairwise) disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$ for all $k \leq n$, if and only if

$$\left| \bigcup_{i \in I} A_i \right| \geq \sum_{i \in I} d_i \tag{*}$$

for all $I \subseteq \{1, \ldots, n\}$. [**Hint:** Construct a bipartite graph in which A is one side, and the other side consists of a suitable number of copies of the sets A_i. Define the edge set of the graph so that the desired result can be derived from the marriage theorem.]

Problem 6. (Exercise 2.11 in Diestel.) Let G be a bipartite graph with bipartition $\{A, B\}$. Assume that $\delta(G) \geq 1$, and that $d(a) \geq d(b)$ for every edge ab with $a \in A$. Show that G admits a complete matching from A to B. [**Hint:** Intuitively, the edges between a set $S \subseteq A$ and $N(S)$ create larger degrees in S than in $N(S)$, so they must be spread over more vertices of $N(S)$ than of S. To make this precise, count both S and $N(S)$ as a sum indexed by those edges. Alternatively, consider a minimal set S violating the marriage condition, and count the edges between S and $N(S)$ in two ways.]