Math 4990 Problem Set 11

Partial solutions and comments

Problem	Points	Mean
Problem 1	2	0.9
Problem 2	2	1.4
Problem 3	2	1.6
Problem 4	2	1.4
$\sum$	8	5.2

Problem 1. IVT applies only to continuous functions. The point is to follow the same proof we had in class, but smooth out the discontinuities that would arise. Most people lost a point for not providing sufficient justification here.

Problem 2. Most people used a thin rectangle as a counter-example (good), but some lost points for not providing enough details.

Problem 3. Most people did this well. Some had solutions that were way more complicated than others.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose it is **periodic**, *i.e.*, there exists $t > 0$ (called the period) such that $f(x) = f(x+t)$ for all x. Show that f has horizontal chords of any length.

Proof. Recall that a continuous function on a closed interval $[0, t]$ has a global min and a global max. Since f is periodic, it has infinitely many of each. Pick $m < M$ such that $f(m) \leq f(x) \leq f(M)$ for all x. Fix $a > 0$ and let $g(x) = f(x) - f(x + a)$. Note that $g(m) = f(m) - f(m + a) \leq 0 \leq f(M) - f(M + a) = g(M)$. As g is continuous, by Intermediate Value Theorem, there exists $c \in [m, M]$ such that $g(c) = 0$, so $f(c) = f(c + a)$ and $[c, c + a]$ is a horizontal chord of length a. $\square$

Incorrect proof. Identify the same m as above, and shift M so that it is in $[m, m + t]$. Note that $f(m) = f(m + t)$ witnesses a chord of length t. Continuously move point x from m to M while keeping it in $[m, M]$. At the same time, continuously move point y from $m + t$ to M while keeping it in $[M, m + t]$. Furthermore, stipulate that $f(x) = f(y)$ at all times. At each point, we get a chord of length $y - x$, which starts from t and goes to 0 continuously. By the IVT, it passes through all values $[0, t]$, which can be added by multiples of t to get all chord lengths. $\square$

The argument breaks down for the following reason: Continuous functions can be very complicated. We do not know that we can move x and y in this fashion towards each other. See, *e.g.*, "Mountain climbing problem" on Wikipedia. Since this is a subtle issue, people who did this received partial credit despite the argument utterly failing to solve the problem at hand.