Math 4990 Problem Set 10

Partial solutions and comments

Problem	Points	Mean
Exercise 6.17	2	1.2
Exercise 6.22	2	1.2
Exercise 6.30	2	1.4
Problem 4	2	1.8
Problem 5	2	1.8
$\sum$	10	7.4

Exercise 6.17. By recalling that each edge belongs to *precisely* two faces from the local topology condition, you can resolve the so-called ambiguity in the picture.

Exercise 6.30.

Proof. For a face f containing a vertex v, denoted $f \ni v$ or $v \in f$, let $a(f, v)$ be the angle of f at v. Let n_f be the number of sides of f. Let V_I and V_B denote the number of vertices in the interior and on the boundary, respectively. Of course, the total number of vertices is given by $V = V_I + V_B$, and the number of edges on the boundary is V_B. Tracking the proof of Theorem 6.25, we see

$$\omega = \sum_{v \in P \setminus \partial P} K(v) = \sum_{v \in P \setminus \partial P} \left(2\pi - \sum_{f \ni v} a(f, v) \right)$$
$$\tau = \sum_{v \in \partial P} K(v) = \sum_{v \in \partial P} \left(\pi - \sum_{f \ni v} a(f, v) \right),$$

where π is substituted for 2π in τ by definition (see page 174). Summing yields

$$\begin{aligned} \omega + \tau &= \sum_{v \in P \setminus \partial P} 2\pi + \sum_{v \in \partial P} \pi - \sum_{v \in P} \sum_{f \ni v} a(f, v) \\ &= 2\pi V_I + \pi V_B - \sum_{f \in P} \sum_{v \in f} a(f, v) \\ &= 2\pi V_I + \pi V_B - \sum_{f \in P} (n_f - 2)\pi. \end{aligned}$$

Now $\sum_{f \in P} n_f$ double counts each edge, **except** for edges on the boundary, each of which is counted only once. As such, $\sum_{f \in P} n_f = 2E - V_B$. Substituting yields

$$2\pi V_I + \pi V_B - \pi((2E - V_B) - 2F) = 2\pi \chi(P),$$

as desired. □