Math 4990 Problem Set 9

Partial solutions and comments

Problem	Points	Mean
Problem 1	3	2.8
Problem 2	3	2.0
Problem 3	7	5.8
$\sum$	13	10.5

Let T_n be the number of domino tilings of a $2 \times n$ region.

Problem 1. Prove that $T_{a+b} = T_a T_b + T_{a-1} T_{b-1}$.

Proof. The LHS is the number of ways to tile a $2 \times (a+b)$ region by dominoes. Consider a tiling of such a region. Let line L cut the region into a $2 \times a$ on the left and a $2 \times b$ on the right. A tiling either has no dominoes crossing L or two dominoes crossing L (why?). There are $T_a T_b$ tilings of the first type and $T_{a-1}T_{b-1}$ tiling of the second type (why?), which gives the RHS. □

Problem 2. Prove that $\binom{n}{1}T_0 + \binom{n}{2}T_1 + \cdots + \binom{n}{n}T_{n-1} = T_{2n-1}$.

Proof. The RHS is the number of ways to tile a $2 \times (2n-1)$ region by dominoes.

For a tiling, count the number of vertical dominoes amongst the left n tiles of the top row. (Obviously the vertical dominoes protrude to the bottom row.) Call this number the *cuteness* of the tiling.

Count the number of tilings with cuteness i. Of the first n tiles on the top row, there are i vertical and $n-i$ horizontal dominoes by definition. They have a combined width of $i + 2(n-i) = 2n - i$. As such, there are $\binom{n}{i}$ ways to tile the first $2n-i$ columns: by deciding which of the n dominoes on the top row are vertical, and then filling the second row with horizontal dominoes in the obvious way. It then remains to tile a region of width $(2n-1) - (2n-i) = i-1$ using dominoes. There are T_{i-1} ways to do so. Thus there are $\binom{n}{i}T_{i-1}$ tilings of cuteness i.

By definition, each tiling has a cuteness between 0 and n. However, cuteness 0 is impossible, as that would mean we have (at least) n horizontal dominoes on the top row, which cannot fit within a region with width $2n-1$. Therefore we conclude that the total number of tilings is $\sum_{i=1}^{n} \binom{n}{i} T_{i-1}$, which is the LHS. □

The most common mistake is to not explain why $i = 0$ is impossible. It is incorrect to use the argument that as $2n-1$ is odd, there is at least 1 vertical tile. Indeed, we are not counting the number of vertical tiles, but vertical tiles amongst the first n tiles of the top row.

Problem 3. Each part is worth a point. Half a point is taken off if there is insufficient justification.