Math 4990 Problem Set 7

Partial solutions and comments

Problem	Points	Mean
Exercise 3.19	2	1.9
Exercise 3.20	2	1.8
Exercise 4.4	2	2.0
Exercise 4.5	2	1.7
Problem 5	2	1.5
Problem 6	2	1.1
$\sum$	12	9.6

Problem 5. Let G be a graph that is maximally planar with at least 4 vertices. Suppose vertices a, b, c are pairwise joined by edges. Show that G has a vertex v distinct from a, b, c such that the degree of v is at most five.

Note that this is a strengthening of Exercise 3.14 we used in class for the proof of Fáry theorem.

Proof. Let G be a graph on $n \geq 4$ vertices. The degree of every vertex is at least 3 (why?).

Assume, towards a contradiction, that such v cannot be found. Then a, b, c each has degree at least 3 and the other $n-3$ vertices each has degree at least six. The sum of the degrees is therefore at least $6(n-3)+9$. On the other hand, the sum of the degrees is twice the number of edges, which is $3n-6$ as shown in class. As such, we get $6n-12 \geq 6n-9$, a contradiction. $\square$

Do not reinvent the wheel, use the bound $3n-6$ proved in class! Most common mistake is claiming that a, b, c each has degree at least 6 as well.

Problem 6. Recall that the number of triangulations of a convex $(n+2)$-gon is the Catalan number C_n. For infinitely many values of n, construct two sets $S, S' \subset \mathbb{R}^2$ each with $n+2$ points such that the number of triangulations of S is greater than C_n and the number of triangulations of S' is nonzero but less than C_n. (See Exercises 3.15 and 3.18.)

Proof. Let us use the "double chain" construction of Exercise 3.18 for S. Suppose there are $m+1$ vertices in each chain. The connecting edges in each chain are always present, but we don't need this fact, as we just need a lower bound on the number of triangulations. Indeed, *suppose* they are present, and count only these triangulations. We may triangulate the top and bottom convex $(m+1)$-gons in C_{m-1} different ways each. The middle portion consists of m upward- and m downward-pointing triangles in any order, so there are $\binom{2m}{m}$ ways to triangulate. As such, there are (at least) $C_{m-1}\binom{2m}{m}C_{m-1}$ triangulations.

We compare this with C_n, where $n+2 = 2(m+1)$ is even. First, recall that $\sum_{i=0}^{t}\binom{t}{i} = 2^t$, so

$$C_n = \tfrac{1}{n+1}\tbinom{2n}{n} < \tbinom{2n}{n} < 2^{2n} = 2^{4m} = 16^m.$$

On the other hand,

$$\frac{\binom{2t+2}{t+1}}{\binom{2t}{t}} = \frac{(2t+2)(2t+1)}{(t+1)(t+1)} = 2(2 - \tfrac{1}{t+1}) \geq 3$$

implies

$$\tbinom{2t}{t} \geq c \cdot 3^t$$

for some absolute constant c. Substituting yields

$$C_{m-1}\tbinom{2m}{m}C_{m-1} \geq c' \cdot \frac{3^m}{m} \cdot 3^m \cdot \frac{3^m}{m} = c' \cdot \frac{27^m}{m^2},$$

where c' is another absolute constant (explicitly, $c' = c^3/9$ works; however, when performing analysis on asymptotics, it's best not to unnecessarily care about the constants). For sufficiently large m, we get

$$C_n < 16^m < c' \cdot \frac{27^m}{m^2} \leq C_{m-1}\tbinom{2m}{m}C_{m-1},$$

as desired. (We can figure out how large m has to be for the inequalities to hold. However, as we only care about "infinitely many values of $n = 2m$," we can stop here and rest assured that all large m works.) $\square$

Most people lost a point for not providing a proof of the inequality.