Math 4990 Problem Set 4

Partial solutions and comments

Problem	Points	Mean
Exercise 1.47	2	1.73
Exercise 1.51	2	1.77
Exercise 1.65	6*	3.09
Problem 4	2	1.45
Problem 5	2	1.73
Problem 6	2	0.70
$\sum$	13	8.93

* Exercise 1.65 is graded out of 6 but is weighted by a factor of 1/2.

Exercise 1.65. Show that a regular tetrahedron cannot be scissors congruent with a cube.

Proof. Let T be a regular tetrahedron of side length x. All its edges have the same dihedral angle $\theta = \arccos \frac{1}{3}$ (why?). We know that $\frac{1}{\pi} \arccos \frac{1}{\sqrt{n}}$ is irrational for $n = 9$ from class, and therefore there exists a $\mathbb{Q}$-linear function $f : V(M_T) \to \mathbb{Q}$ such that $f(\pi) = 0$ and $f(\theta) = 23$. For this f, the corresponding Dehn invariant $D_f(T) = 6 \cdot x \cdot 23$ is nonzero, but any Dehn invariant of the cube is 0, and hence by Dehn–Hadwiger theorem, T is not scissors congruent to a cube. □

Note that there are plenty of functions whose corresponding Dehn invariants are 0 for T. Therefore, to use Dehn–Hadwiger theorem, you should *pick* a specific function f. The reason you can do it is because θ is not a rational multiple of π. In particular, you should

(1) show *some* work for calculating θ,
(2) show that θ is not a rational multiple of π (perhaps by using a theorem from class),
(3) conclude the existence of a suitable $\mathbb{Q}$-linear function f,
(4) whose associated Dehn invariant is nonzero for T,
(5) mention that Dehn invariants for the cube are always zero, and
(6) cite Dehn–Hadwiger theorem (not Sydler theorem!) to finish the proof.

Problem 4. Two polygons are **∂-congruent** if one can be decomposed into polygonal pieces and rearranged to form the other, such that the boundary points remain on the boundary. Prove that polygons of the same area and perimeter are ∂-congruent.

Note that it is *not* true that a polygon is ∂-congruent to a rectangle of the same area and perimeter. Indeed, let C be a regular $10^{1000000000}$-gon with perimeter 2π and let R be a rectangle with side lengths x and y so that $2(x + y) = 2\pi$. The area of C is roughly π while R has area at most $\pi^2/4 < \pi$. So going down that route does not work.

Problem 6. Take an n-gon, a translated copy of it (not in the same plane), and join corresponding vertices by edges to form n new faces (necessarily parallelograms). This polyhedron with $n + 2$ faces is called a **prism**. Prove that prisms are scissors congruent to cubes (of the same volume). Do not assume that the prisms are *right* prisms, *i.e.*, they might be slanted.

There are two ways of proving this. Let's call the two n-gons **base** and the others **side**.

Using Sydler theorem. One would calculate *all* Dehn invariants corresponding to every single $\mathbb{Q}$-linear f and show that they always vanish, just like that of the cube. For this proof, there are two points regarding non-right prisms. The first is that the edges of the base polygons do not have $\pi/2$ as dihedral angles. Most of you correctly dealt with this. The second is that the dihedral angles of the n side edges joining the base polygons together are **not** the same as the angles of the base n-gon. Those who did not address this lost a point.

Cut with scissors explicitly. Some tried to use a dissection of the base polygon to form small prisms, each of which has a small polygon as its base. These prisms all have the same height, so it is reasonable to try rearranging them into a different prism, perhaps by using the 2-dimensional rearranging of the base into some shape (rectangle is a popular choice) as afforded by Bolyai–Gerwein theorem. However, the plight of the non-right prism strikes again! If the rearranging included rotations (in the plane of the base), the prisms, not being right, are now slanting in all sorts of weird directions. What you'll need to do is rearrange without using rotation (*à la* T-congruence). This does not always work. One way is to start from the beginning and make a cut that is orthogonal to the side edges (not parallel to the base) and rearrange the two pieces into a right prism (with a different base!) before proceeding.

Exercise 1.48. Assume no three vertices of a polygon P are collinear. Prove that out of all possible dissections of P into triangles, a triangulation of P will always result in the fewest number of triangles.

This is false. A triangulation of a polygon with n vertices has $n-2$ triangles. Figure 1 gives a counterexample with $n=6$ vertices and a dissection with $n-3$ triangles.

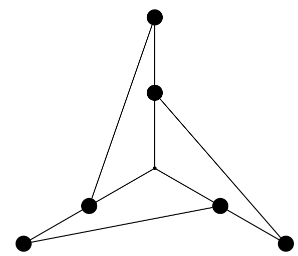

FIGURE 1. A simple counterexample to Exercise 1.48.

Note that the "generic" condition of "no collinear vertices" is not strong enough. Here there are three *concurrent* edges (when extended). However, even if we add "no concurrent edges" to the generic condition, it is still false. Figure 2 shows a counterexample with $n=10$ vertices.

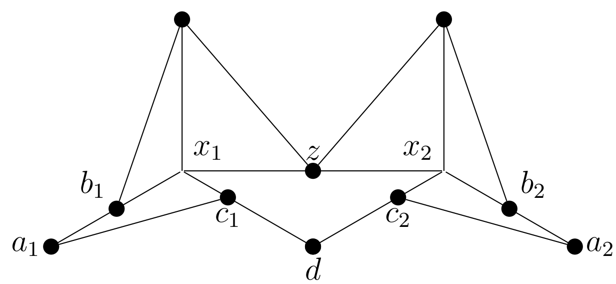

FIGURE 2. A more complicated counterexample to Exercise 1.48.

This example is still not fully generic. Indeed, note that x_1 is the intersection of the (extended) edges a_1b_1 and c_1d. As such, one could write the coordinates of x_1 in terms of those of a_1, b_1, c_1, d. Similarly, x_2 is determined by a_2, b_2, c_2, d. As x_1, z, and x_2 are collinear, the vertices $a_1, a_2, b_1, b_2, c_1, c_2, d, z$ satisfy some kind of non-trivial relationship.

This shows that there could be complicated hidden relationships amongst the vertices. One way to circumvent all this is to require that no vertex be a convex combination of the other vertices.

Indeed, suppose P is a *convex* polygon on n vertices, and consider an arbitrary dissection. Sum the angles of the triangles in the dissection to get $t\pi$, where t is the number of triangles. On the other hand, sum only those angles occurring at vertices of P. This yields $(n-2)\pi \leq t\pi$, as desired.

The key point used here is that P has no reflex angles, so it cannot be dissected in such a way as to have a ~~thorn~~ vertex in a triangle's side.