Math 4990 Problem Set 3

Partial solutions and comments

Problem	Points	Mean
Exercise 1.30	2	1.74
Exercise 1.31	2	1.87
Exercise 1.35	2	1.78
Exercise 1.36	2	1.87
Exercise 1.39	2	0.59
Exercise 1.40	2	0.83
$\sum$	12	8.67

Exercise 1.30. Modify Lemma 1.18 to show that one guard placed anywhere in a convex polygon can cover it.

Proof. Assume P is convex but there are a pair of **points** a and b in P **such that the segment** ab **does not lie within** P. We identify a reflex vertex of P to establish the contradiction. Let σ be the shortest path connecting a to b entirely within P. It cannot be that σ is a straight line segment contained inside P, for then **segment** ab **is inside** P. Instead, σ must be a chain of line segments. Each corner of this polygonal chain turns at a reflex vertex—if it turned at a convex vertex or at a point interior to P, it would not be the shortest. $\square$

Note that only a few (**bolded**) words need to be changed.

Exercise 1.35. It is acceptable to say that an ear may not exist. However, it would be better to say that triangulations cannot (always) be properly coloured using 3 colours. Indeed, the latter is the real reason why the colouring argument does not identify a set of $n/3$ vertices. Generic claims should be backed up by evidence (*e.g.*, counter-examples).

Exercise 1.39. To cover the *exterior* of polygons with n vertices, $\lceil n/2 \rceil$ boundary guards are needed for some polygons, and sufficient for all of them.

Proof. Let P be an n-gon. If P is convex, then $\lceil n/2 \rceil$ guards is necessary (why?) and sufficient (why?).

Now suppose P is not convex. It remains to show that $\lceil n/2 \rceil$ is sufficient. By (the desired version of) Exercise 1.13,[1] polygon P has a mouth, that is, there is a diagonal that is *completely* external. Draw this diagonal and consider P with this new triangle added a bigger polygon. Continue until we get a convex polygon Q. (Note that Q is the *convex hull* of P.) Note that the polygonal regions inside Q but outside P is triangulated by our process. Pick a vertex q of Q (which is also a vertex of P) and split it into two vertices q_1 and q_2, in such a way as to open up a passage from inside P to the outside (how?). Add a vertex v and connect it to each vertex of Q (including q_1 and q_2 but not q) by a possibly curved segment, in such a way as to keep what was inside P still part of the exterior (how?). The resulting graph admits a proper 3-colouring (why?). Fix such a colouring. Of the two colours not used at v, the one used less frequently is used at most $\lceil n/2 \rceil$ times (why?). Put guards there. They cover the region inside Q and outside P (why?). Moreover, since every other vertex of Q has a guard (why?), the exterior of Q is covered (why?). $\square$

[1] A non-convex polygon has a mouth.

One part is to show that $\lceil n/2 \rceil$ guards are sufficient for all polygons. Some people only guarded the boundary, not the infinite exterior. It is "obvious" that placing a guard on every other vertex guards the boundary. This may fail to guard the exterior! (Turn your answer for Exercise 1.31 inside out.) In fact, if your proof ends up putting a guard on every other vertex, you got no credit since it does not work.

To show that $\lceil n/2 \rceil$ guards are sometimes necessary, you should pick a polygon for every n. It is not enough to just do it for, say, the $n = 3$ case. Indeed, think about the "comb" we had in class or what you did for Exercise 1.40. If you did it only for a few cases, you got half credit.

Exercise 1.40. There are several ways of doing this. The point is to do it for all values of n. If you did the problem for infinitely many but not all values of n, you got half credit.