Math 4990 Problem Set 2

Partial solutions and comments

Problem	Points	Mean
Exercise 1.21	2	1.8
Exercise 1.22	2	0.5
Problem 3	2	1.1
$\sum$	6	3.4

Exercise 1.22. For any $n \geq 3$, show there is no polygon with $n+2$ vertices with exactly $C_n - 1$ triangulations.

Proof. Let P be an $(n+2)$-gon.[1] If P is convex, it has C_n triangulations by Theorem 1.19.[2] Therefore, we may assume P is not convex. By Lemma 1.18, this means there is a pair of nonadjacent vertices u and v such that the line segment uv is *not* a diagonal of P. Following the proof of Theorem 1.20,[3] let Q be a convex polygon with $n+2$ vertices, labelled the same way as P. Each diagonal of P corresponds to a diagonal of Q, and if two diagonals of P do not cross, neither do they cross in Q. So every triangulation of P determines a triangulation of Q.

Let $\mathcal{T}(P)$ denote the set of triangulations of P, and similarly $\mathcal{T}(Q)$ for Q. The above correspondence gives an injection (a one-to-one map) $f : \mathcal{T}(P) \to \mathcal{T}(Q)$. Let $M \subseteq \mathcal{T}(Q)$ be the set of triangulations of Q where the diagonal uv is used. Since uv is not a diagonal of P, M is not in the image of f, so f is an injection from $\mathcal{T}(P)$ to $\mathcal{T}(Q) \setminus M$,[4] so we have

$$|\mathcal{T}(P)| \leq |\mathcal{T}(Q) \setminus M| = |\mathcal{T}(Q)| - |M|.$$

It remains to show[5] that $|M| > 1$.

The diagonal uv divides Q into two convex polygons Q_1 and Q_2. Let Q_1 have at least as many vertices as Q_2. Since Q is an $(n+2)$-gon, Q_1 has at least 4 vertices and hence at least $C_2 = 2$ triangulations. These triangulations extend to different triangulations in M by taking any maximal set of noncrossing diagonals, so $|M| > 1$, as desired. □

Many people proved that if there is one reflex angle, at least two triangulations are "killed," and used the intuition that introducing more reflex angles would only make it worse. This is way too informal and requires justification.

Showing that "not using a diagonal" kills at least two triangulations is a main point of this problem. Thus it is not acceptable to state it without proof.

Without following the proof of Theorem 1.20, it is rather difficult to make precise what it means for triangulations to be "killed." Indeed, why wouldn't moving vertices introduce more triangulations? Note that not introducing more *diagonals* does not immediately imply no more triangulations, since there is a very important requirement that the diagonals are *noncrossing.*

[1] It is better to write "$(n+2)$-gon" instead of "$n+2$-gon."

[2] Get in the habit of citing theorems!

[3] Use proofs in the textbook for inspiration.

[4] If A and B are sets, $A \setminus B$ is the set of elements in A but not in B.

[5] "It remains to show" is a very useful phrase in reminding us that we have reduced the original problem to a new, hopefully easier, problem.

Indeed, perhaps making more reflex vertices would allow existing diagonals that were crossing to become noncrossing.

This leads to another error committed by some students. Some suggested allowing diagonals to be "exterior" and consider these generalized class as "exterior triangulations." How do you define these? The most natural way seems to be to take a maximal set of noncrossing diagonals (possibly exterior). However, this does not preserve the number of diagonals, and hence makes comparing them to triangulations of a convex polygon (in order to relate to C_n) difficult. Another way is to call them noncrossing if and only if they are noncrossing in a corresponding convex polygon. This certainly requires details and definitions, and possibly some proofs about their properties. No one going down this route was able to supply enough details to make the argument convincing.

Problem 3. Let $\mathcal{B}_{n+1}$ be the set of binary parenthesizations of a string of $n+1$ letters. For example,

$$\mathcal{B}_4 = \{((xx)x)x, x((xx)x), (x(xx))x, x(x(xx)), (xx)(xx)\}.$$

Establish a bijection between $\mathcal{B}_{n+1}$ with either $\mathcal{T}_{n+2}$ or $\mathcal{P}_n$ to conclude that $|\mathcal{B}_{n+1}| = C_n$.

Proof. We first define a map $f : \mathcal{B}_{n+1} \to \mathcal{P}_n$.[6] Ignoring $\boxed{(}$, turn each letter into a right step and each $\boxed{)}$ an up step. Note that we always start with two right steps; drop one of them. Add an extra up step at the end. This is certainly a monotonic path from $(0,0)$ to (n,n) (why?). We must check that this path does not cross above the diagonal.[7] In other words, when we encounter the ith $\boxed{)}$, we must have already encountered at least $i+1$ letters.

Consider the following process, which involve $n+1$ apples.[8] We scan a binary parenthesization from left to right. Each time we see a letter, put an apple on a table in its own **pile**. Each time we see a $\boxed{)}$, merge the last two piles together. After we encounter the ith $\boxed{)}$, we merged a total of i times. There must be at least one pile on the table at this point.[9] Backtrack and unmerge the piles (without removing the apples). Each unmerge creates one more pile, so we now have at least $i+1$ piles. Each pile has at least one apple. So we must have placed at least $i+1$ apples. This means we must have encountered at least $i+1$ letters. In other words, the monotonic path does not cross above the diagonal, and the map f is well-defined.

We now define an inverse map $g : \mathcal{P}_n \to \mathcal{B}_{n+1}$.[10] It is obvious how to reverse f to turn a path to a sequence of letters and $\boxed{)}$. It remains to insert $\boxed{(}$ in the correct places. Let us consider the apple merging process again with some modifications. When placing apples, put them from left to right. Imagine that they are very heavy, so we do not want to move them. To indicate that we are merging two piles, we simply draw a circle surrounding the two piles (and nothing else). Since the path does not cross above the diagonal, we always have at least two piles when we are asked to merge, so this process is well-defined. We end up with $n+1$ apples on a table and $n-1$ circles drawn around them. This is ridiculous. Take a picture with a weird aspect ratio so the apples barely fit in the frame. Since the top and bottom parts of the circles are cut off, they look like parentheses! Yay. By construction, $f \circ g$ and $g \circ f$ are the identity functions on $\mathcal{P}_n$ and $\mathcal{B}_{n+1}$, respectively, so $|\mathcal{B}_{n+1}| = |\mathcal{P}_n| = C_n$, as desired. □

[6] Most people opted to use $\mathcal{P}_n$, which is arguably easier. (Going via triangulations is harder. Most people were unable to do this correctly. The analogous check for the "crossing above the diagonal" here is to check that the diagonals drawn in the triangulation are noncrossing.)

[7] This is crucial. Recall that we took great pains to calculate the size of $\mathcal{P}_n$, whereas the monotonic paths are much easier to count. Such omission is considered a major error and costs a point.

[8] As the saying goes, an apple a day, keeps the doctor (of philosophy) away.

[9] For there are apples on the table—we are not allowed to eat them!

[10] Another common error is forgetting to exhibit an inverse or sufficiently define the inverse map.