# Math 4990 Problem Set 1

*Partial solutions and comments*

**Exercise 1.5.** One possible proof is to use induction on the number $h$ of holes. Note that it is unnecessary to start the induction with $h = 1$. You should start it with $h = 0$ and cite Theorem 1.4 as the base case. "Do not reinvent the wheel."

For the induction step, one can draw a diagonal from a vertex on the boundary to a vertex of a hole. One should explain why this is possible! (Those who did not lost a point.)

Also, technically, the boundary of a polygon needs to be a simple curve. Counting a drawn diagonal as "two edges" does not make it a polygon. One should say a few words about this. One rigorous way is to define a more general class called "polygons with holes and diagonals."

Another way to do it rigorously is to "cut through" and break the region into two regions with fewer holes (but not necessarily no holes). Indeed, many people tried to do the following. Order the holes $H_1, \ldots, H_h$. Draw diagonals from $P$ to $H_1$, from $H_i$ to $H_{i+1}$ for $i = 1, \ldots, h-1$, and $H_h$ to $P$. These $h + 1$ diagonals break the region into two polygons (with no holes). Unfortunately, this is not always possible.

**Exercise 1.14.** If you follow the proof I gave in class, the equations are $e = n + d$ (easy), $f = 1 + h + t$ (obvious), $2e = n + 3t$ (double counting), and $f = 2 + d$ (same, but requires new explanation—not obvious).

Several students supplied the following proof, which I had not seen before. I like it very much:

*Proof.* An $n$-gon has interior angles summing up to $(n - 2)\pi$ (why?) and exterior angles summing up to $(n + 2)\pi$ (why?). A polygon $P$ with $h$ holes and $n$ total vertices has interior angles summing up to $(n - 2 + 2h)\pi$ (why?). Each triangle has interior angles summing up to $\pi$. So if $P$ is triangulated, there are necessarily $n - 2 + 2h$ triangles. □

It is worth noting that I graded Exercises 1.5 and 1.14 together, in the sense that if you make a (fixable) mistake in both 1.5 and 1.14, I did *not* take off points twice. Think of 1.14 as "assuming you did 1.5 reasonably, how would you calculate the numerics?" I tried to grade that as fairly as possible depending on the claims you made in 1.5.

**Exercise 1.10.** Prove by induction that every polygon with more than three vertices has at least two ears.

*Proof.* We prove the stronger statement by induction on the number of vertices: "Every polygon with more than three vertices has at least two ears *with non-adjacent tips*."

Let $P$ be an $n$-gon, $n \geq 4$. By Lemma 1.3, $P$ has a diagonal $xy$, which cuts $P$ into $P_1$ and $P_2$.

If $P_1$ is a triangle, let $b$ be the vertex not on $xy$. Otherwise, as $P_1$ has fewer vertices than $P$, by the induction hypothesis, $P_1$ has two ears with non-adjacent tips $b_1$ and $b_2$. As they are not adjacent, at least one of the tips is not on $xy$.

In either case, we found an ear $a, b, c$ such that $b$ is not on $xy$. The only edge of $P_1$ not in $P$ is $xy$, so $ab$ and $bc$ are both edges of $P$. Moreover, if $ac$ is a diagonal of $P_1$ then it is also a diagonal of $P$. This shows that $a, b, c$ is an ear of $P$.

Similarly, $P_2$ contains an ear with tip $b'$ not on $xy$. Note that $b$ and $b'$ are non-adjacent ear tips of $P$, as desired. □

Some of you tried using induction on the original statement without strengthening the induction hypothesis to require non-adjacency of the tips. As far as I know, no such proof works. Therefore, anyone not following the *non-adjacent tips* tip lost **all points** for this problem. If you think you have a working inductive proof without this strengthening, do not hesitate to come talk to me.

Many people tried starting at a convex vertex $b$, whose neighbours $a$ and $c$ might admit a diagonal $ac$. In that case, an ear has been found. Otherwise, find a vertex $d$ such that $bd$ is a diagonal. This requires re-doing the "sweeping" proof of Lemma 1.3. Note that this is bad **style**.[1] Instead of invoking the lemma, you are proving the lemma again in this special case. Please *do not reinvent the wheel.* Cite the lemma and move on.

Another common mistake is to delete or add a vertex. The point of Corollary 1.9 is for us to find nice vertices to delete in order to do induction geometrically. (This is written in the paragraph above.) As such, you are not allowed to do this during the proof! (The math police will give you a citation for circular reasoning.)

Similarly, some people copied the proof in the textbook, possibly by using induction in a non-essential way. This is not acceptable, as clearly the point of Exercise 1.10 is to give an alternate proof. It is a good idea to figure out the point of a problem so as to not make this kind of interpretation errors.

**Exercise 1.13.** For a theorem about the *existence* of mouths, I am looking for something analogous to Corollary 1.9, namely, that something *has* mouths.

Most proved something (equivalent to) "a convex polygon has no mouths" and some proved "a polygon cannot have too many mouths." These are arguably about the *non-existence* of mouths and so I gave only half credit.

[1] Good mathematical style is hard to teach or define, and probably can only be learned with experience. I will try to point this out to you when the offense is obvious to most mathematicians. Reinventing the wheel is one of the main indicators. The other is unnecessarily long proofs.