Math 4990 Problem Set 12

Due Tuesday, Dec 08, 2015 in class

Assignment

The Envy-Free Division lecture is based on [Su] (see link on website), which you are welcome to peruse.

Problem 1. Provide the missing component in the following analogy:

$$\text{Sperner's Lemma : envy-free cake division :: \_\_\_\_\_\_\_ : envy-free rent division}$$

Namely, take a triangle with vertices A, B, C, add one or more vertices on each edge, add vertices in the interior, and then triangulate. Label vertex A with 2 or 3, B with 1 or 3, and C with 1 or 2. Label vertices on the interior of edge AB with 3, AC with 2, and BC with 1. Label the vertices inside with 1, 2, or 3. A small triangle is **rainbow** if its vertices all have different labels.

Prove that the number of rainbow triangles is odd.

Problem 2. Suppose k pirates found a necklace made up of t types of pearls, where the number of each type of pearls is a multiple of k. The necklace is a linear chain (not circular). They wish to cut the necklace to divide the loot evenly, *i.e.*, each pirate receives $\frac{1}{k}$ proportion of each type of pearls.

(1) Prove that $t(k-1)$ cuts may be needed.
(2) Prove that $t(k-1)$ cuts always suffice.

[**Hint:** We did the case of $t = 2$ and arbitrary k. If you have difficulty with this problem, you may assume $t = 3$ for partial credit. You may, moreover, assume $k = 2$ for even less partial credit.]

Problem 3. Let $f, g : [0, 1] \to [0, \infty)$ be two continuous functions such that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = 1.$$

Prove that for every $n \in \mathbb{N}$, there exist $a, b \in [0, 1]$ such that

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx = \tfrac{1}{n}.$$

[**Hint:** For partial credit, do the case where f and g are *strictly* positive.]