Math 4990 Problem Set 9

Due Tuesday, Nov 10, 2015 in class

Please refer to previous problem sets for instructions, including but not limited to the collaboration policy.

Assignment

Let T_n be the number of domino tilings of a $2 \times n$ region. Recall the *linear recurrence relation*

$$T_n = T_{n-1} + T_{n-2} \qquad (*)$$

for $n \geq 3$. Using (*) for $n = 2$ suggests that it is sensible to define $T_0 = 1$. For the rest of the problem set, it is advisable to **NOT** use (*). That is, each time you see T_n, interpret it as "number of tilings" as opposed to the actual number.

Problem 1. Prove that $T_{a+b} = T_a T_b + T_{a-1} T_{b-1}$.

Problem 2. Prove that $\binom{n}{1} T_0 + \binom{n}{2} T_1 + \cdots + \binom{n}{n} T_{n-1} = T_{2n-1}$.
[**Hint:** Associate a number i to each domino tiling of a $2 \times (2n-1)$ region, $1 \leq i \leq n$, such that there are $\binom{n}{i} T_{i-1}$ tilings associated with i.]

Problem 3. Let $f(n)$ denote the number of domino tilings of a $3 \times n$ region, and let $g(n)$ denote the number of domino tilings of the same region but with one corner square removed.

(1) Calculate $f(n)$ and $g(n)$ from their definitions for $1 \leq n \leq 4$.
(2) Write $f(n)$ in terms of $f(a)$ and $g(b)$ for some $a, b < n$.
[**Hint:** Follow the idea that allowed us to find (*).]
(3) Similarly, write a recurrence relation for $g(n)$ in terms of f and g.
(4) Using the recurrence relations, define $f(0)$ and $g(0)$ sensibly.
(5) Obtain a linear recurrence relation of f alone by eliminating g.
(6) Let $h(n) = f(2n)$ and calculate $h(n)$ for $0 \leq n \leq 9$.
[**Hint:** One way to do this is by using Wolfram Alpha. For example, to get the first ten Fibonacci numbers, go to `http://www.wolframalpha.com`, enter

```
f(n)=f(n-1)+f(n-2), f(0)=0, f(1)=1
```

and click "more" on the result page.]
(7) What is the number of domino tilings of a 3×40 region?
[**Hint:** One way to do this is by using the On-Line Encyclopedia of Integer Sequences. Go to `http://oeis.org` and search with the first few terms of the sequence.]