Math 4990 Problem Set 5

Due Tuesday, Oct 13, 2015 in class

Refer to previous problem sets for instructions, including but not limited to the collaboration policy.

Read the assignment carefully. Some problems are similar but not identical to those in the book.

Assignment

Liberally peruse **pages 33–36** of [DO].

Problem 1. Recall these definitions from class:

A set $M \subseteq \mathbb{R}^d$ is **convex** if for every pair $x, y \in M$, the line segment from x to y lies in M.

For a set $S \subseteq \mathbb{R}^d$, the **convex hull** of S, denoted $\text{conv}(S)$, is the intersection of all convex sets that contain S.

Prove that the convex hull $\text{conv}(S)$ of any set S is convex.

(Do not assume that S lies in the plane. Do not use Theorem 2.2, as its proof relies on this exercise.)

Problem 2. Let $S \subset \mathbb{R}^d$ be a finite point set with at least four points. For $d = 2$, show that S can be partitioned into two sets A and B such that $\text{conv}(A)$ intersects $\text{conv}(B)$. (Do not use Helly theorem, as this fact is used in its proof.) Does the result hold for $d \geq 3$? (As always, provide justification.)

Problem 3. Let $S \subset \mathbb{R}^2$ be a finite point set in the plane. Show that $\text{conv}(S)$ is the convex polygon with the smallest perimeter that contains S.

Problem 4. Let $S \subset \mathbb{R}^3$ be a finite point set in space. Show that $\text{conv}(S)$ is the convex polyhedron with the smallest volume that contains S.

Problem 5. Let $P_1, \ldots, P_n \subset \mathbb{R}^2$ be rectangles whose sides are parallel to the x- and y-axes. Show that if every *two* of them intersect then they all intersect, *i.e.*, there is a point $z \in P_1, \ldots, P_n$.

Problem 6. Recall this corollary of Helly theorem, stated in class:

> Let $A \subset \mathbb{R}^2$ be a fixed convex set and let $X_1, \ldots, X_n \subset \mathbb{R}^2$ be convex sets such that every three of them intersect a translation of A. There exists a translation of A that intersects all sets X_i.

For each i, let $Y_i = \{y \in \mathbb{R}^2 : (A + y) \cap X_i \neq \varnothing\}$, where $A + y := \{a + y : a \in A\}$ is the translation of A by y. In order to apply Helly theorem to obtain the corollary, show that the Y_i are convex.

Optionally, for no credit, explain whether it is possible to drop either one of the word "convex" from the statement of the corollary.