Math 4990 Problem Set 2

Due Tuesday, Sep 22, 2014 in class

Instructions

As before, write your solutions neatly and clearly, with complete justification (*i.e.*, proof). Please don't submit scratchwork. Consider using LaTeX. Draw figures (possibly by hand) whenever suitable.

Collaboration is encouraged, as long as you

(1) understand your solutions,
(2) write your solutions in your own words without copying, and
(3) **indicate the names of your collaborators**.

Unless otherwise specified, all numberings refer to the textbook [DO].

I wanted to provide you feedback on your first assignment before you do too many more problems (possibly repeating certain stylistic errors from the first homework). As such, I am giving you a short second problem set. Please do not expect all future ones to be of this length.

Assignment

Liberally peruse **pages 8–13** of [DO].

Read the lecture wrap-up section below (on the second page).

Do **Exercise 1.21** (remember, as always, to justify your answer) and **Exercise 1.22** ("for each" and "for any" means the same thing here, you should prove this for *all* $n \geq 3$).

Problem 3. Let $\mathcal{B}_{n+1}$ be the set of binary parenthesizations of a string of $n+1$ letters. For example,

$$\mathcal{B}_4 = \{((xx)x)x, x((xx)x), (x(xx))x, x(x(xx)), (xx)(xx)\}.$$

Establish a bijection between $\mathcal{B}_{n+1}$ with either $\mathcal{T}_{n+2}$ or $\mathcal{P}_n$ to conclude that $|\mathcal{B}_{n+1}| = C_n$.

Problem 4. Optionally, for no credit, pick yet another family of Catalan objects (see below) and establish a bijection to one of the three families we already considered. Make sure to define the objects being considered.

Lecture wrap-up

Catalan numbers background. On the course website is a link to a webpage full of links about Catalan numbers curated by Igor Pak. That webpage includes historical details about Catalan numbers as well as mathematical articles. It may be fun to take a look at the links to Richard Stanley's "Exercise 19" and "Catalan addendum"—totalling more than 200 families of objects counted by Catalan numbers. Triangulations $\mathcal{T}_{n+2}$ is (a), the first one listed. The paths $\mathcal{P}_n$ we discussed is (h) and the parenthesizations in Problem 3 is (b). You can pick one from here for (the optional) Problem 4.

Alternate proof. In class, using a series of problems, we proved Theorem 1.19 together. Note that the textbook contains a different proof. This was in your reading last week and re-assigned this week. Please review it if you have not read it.

Convexity. At the end of the proof, I was asked where we used convexity. This is a very important question! Any time we finish a proof, it is a good practice to check if we used all the hypotheses. For this case, we are using the fact that a diagonal exists between any two nonadjacent vertices of a convex polygon. This is Lemma 1.18 (again in your reading assignment last week).

Well-definedness. At the end of the class, I was a bit rushed to explain why the inverse map $\mathcal{P}_n \to \mathcal{T}_{n+2}$ is well-defined. To explain, let me first describe the map again.

Let a path from $\mathcal{P}_n$ be given. Label the right steps $3, 4, \ldots, n+2$. For each $v = 3, 4, \ldots, n+2$, in that order, consider the up steps immediately following the corresponding right step. For each up step, pick the largest u such that $1 \leq u < v$ and uv can be drawn as a diagonal without crossing other diagonals already drawn. Draw diagonal uv. This is the map.

In general, we say a map is *well-defined* if we can always follow its procedure without getting stuck. In this case, we need to check that each time we look for a u in the process above, we must be able to find at least one (so we can use the largest such u).

More specifically, consider the step when $v = i$. We are asked to draw at most $i-2$ diagonals in total (why?). Therefore, suppose we have drawn at most $i-3$ diagonals and are asked to draw one more. Let us consider whether this can be done.

From vertices 1 to i, we have $i-1$ edges, and at most $i-3$ diagonals. Choose a point q on the edge between vertices 1 and $n+2$ of the polygon. Each diagonal drawn creates a triangle on the "far side." That is, from the vantage point of q, each time a diagonal is drawn, two line segments (edge or diagonal) are obscured by this new diagonal. Conversely, a diagonal can be drawn as long as there are at least 2 line segments remaining. We start with $i-1$ line segments (all edges) and decrease the count by at most $i-3$, one for each diagonal drawn, to obtain at least 2 line segments remaining. This means that we can draw another diagonal, as desired. $\square$