Math 4990 Scissors congruence and Dehn invariants

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§1. $\mathbb{Q}$-vector spaces. For $M = \{m_1, \ldots, m_k\} \subseteq \mathbb{R}$, let

$$V(M) := \Big\{ \sum_{i=1}^{k} q_i m_i : q_i \in \mathbb{Q} \Big\} \subseteq \mathbb{R}$$

denote the set of all linear combinations of numbers in M with rational coefficients.

Note that $V(M)$ is a finite-dimensional vector space over the field $\mathbb{Q}$.

The dimension is the size of any minimal generating set. As M generates,

$$\dim_{\mathbb{Q}} V(M) \le k = |M| .$$

Considering $\mathbb{Q}$ as a 1-dimensional vector space over itself, a linear map

$$f : V(M) \to \mathbb{Q}$$

of $\mathbb{Q}$-vector spaces is a $\mathbb{Q}$-linear function, which satisfies these properties:

(i) $f(x) + f(y) = f(x + y)$ for $x, y \in V(M)$
(ii) $f(qx) = qf(x)$ for $x \in V(M)$, $q \in \mathbb{Q}$

§2. Dehn invariants. Let P be a 3-dimensional polyhedron. For an edge e of P, let $\ell(e)$ denote its length, and $\phi_P(e)$ its **dihedral angle**, defined as the angle between the two faces meeting at e. (If n_1 and n_2 are *outward* unit normal vectors of the faces in question, then $n_1 \cdot n_2 = -\cos(\phi_P(e))$. [DO] is missing a negative sign.)

Let M_P be the set of all dihedral angles of P and π.

Example 1 (1.58)**.** Cube C has $M_C = \{\pi, \pi/2\}$. Let T be the standard simplex (see [DO] Fig 1.26b). $M_T = \{\pi, \pi/2, \arctan\sqrt{2} = \arccos\frac{1}{\sqrt{3}}\}$.

Definition 2. For a $\mathbb{Q}$-linear function $f : V(M) \to \mathbb{Q}$ with $f(\pi) = 0$, define the **Dehn invariant** $D_f(P)$ of P (with respect to f) by

$$D_f(P) := \sum_{e \in P} \ell(e) \cdot f(\phi_P(e)),$$

where the sum runs over all edges e of the polyhedron P.

[DO] calls such a function where $V(M) = \mathbb{R}$ a *d***-function**. This approach requires considering $\mathbb{R}$ as an infinite-dimensional $\mathbb{Q}$-vector space, which might be harder to swallow.

For any such f, as $f(\pi/2) = \frac{1}{2} f(\pi) = 0$, we have $D_f(C) = 0$.

§3. Proof of Dehn–Hadwiger theorem [AZ§8].

Theorem 3 (Dehn–Hadwiger 1.62)**.** *Let P be a polyhedron decomposed into finitely many polyhedral pieces $P_1, \ldots, P_k$. Let $f : V(M) \to \mathbb{Q}$ be $\mathbb{Q}$-linear with $f(\pi) = 0$, where $M \subset \mathbb{R}$ is finite and*

$$M \supseteq M_P \cup M_{P_1} \cup \cdots \cup M_{P_k}.$$

Then

$$D_f(P) = \sum_{i=1}^{k} D_f(P_i).$$

Proof. Let S denote all edge segments. For a polyhedron Q and a straight line segment s, define the dihedral angle as follows. If s is part of an edge e, then it shares the dihedral angle of e. If s lies on a face or in the interior, then the dihedral angle is π or 2π, respectively. Otherwise, say it is 0.

The first key observation is that

$$\phi_P(s) = \sum_i \phi_{P_i}(s), \tag{*}$$

for any $s \in S$, regardless of its spatial relationships to P and the P_i.

The second key observation is that Dehn invariants can be calculated over (all) edge segments:

$$\begin{aligned} D_f(Q) &= \sum_{e \in E(Q)} \ell(e) \cdot f(\phi_Q(e)) \\ &= \sum_{e \in E(Q)} f(\phi_Q(e)) \cdot \sum_{\substack{s \in S \\ s \subseteq e}} \ell(s) \\ &= \sum_{e \in E(Q)} \sum_{\substack{s \in S \\ s \subseteq e}} \ell(s) f(\phi_Q(s)) \\ &= \sum_{s \in S} \ell(s) f(\phi_Q(s)). \end{aligned} \tag{†}$$

The rest is a simple calculation:

$$\begin{aligned}
\sum_i D_f(P_i) &= \sum_i \sum_{e\in E(P_i)} \ell(e)\cdot f(\phi_{P_i}(e)) \\
&= \sum_i \sum_{s\in S} \ell(s)\cdot f(\phi_{P_i}(s)) && \text{by } (\dagger) \\
&= \sum_{s\in S} \sum_i \ell(s)\cdot f(\phi_{P_i}(s)) \\
&= \sum_{s\in S} \ell(s)\cdot \sum_i f(\phi_{P_i}(s)) \\
&= \sum_{s\in S} \ell(s)\cdot f\Big(\sum_i \phi_{P_i}(s)\Big) && \text{by } \mathbb{Q}\text{-linearity} \\
&= \sum_{s\in S} \ell(s)\cdot f(\phi_P(s)) && \text{by } (*) \\
&= D_f(P), && \text{by } (\dagger)
\end{aligned}$$

as desired. □

Corollary 4 (1.62)**.** *Let P and Q be two polyhedra, and $M \subset \mathbb{R}$ finite such that $M \supseteq M_P \cup M_Q$. If there exists a $\mathbb{Q}$-linear function $f : V(M) \to \mathbb{Q}$ with $f(\pi) = 0$ such that $D_f(P) \neq D_f(Q)$, then P and Q are not scissors congruent.*

Proof. Suppose P and Q are scissors congruent, and let (M and) f be given.

Fix some common decomposition: P is decomposed into $P_1, \ldots, P_k$ and Q is decomposed into $Q_1, \ldots, Q_k$, where P_i and Q_i are congruent. Let $M' \supseteq M$ be a finite set that includes all dihedral angles that appear.

Extend f to $f' : V(M') \to \mathbb{Q}$ by specifying the values of f' on new basis elements (and keep the old ones the same). Then

$$D_f(P) = D_{f'}(P) = \sum_{i=1}^{k} D_{f'}(P_i) = \sum_{i=1}^{k} D_{f'}(Q_i) = D_{f'}(Q) = D_f(Q),$$

as desired. □

Example 5 (1.64)**.** Recall that cube C has $M_C = \{\pi, \pi/2\}$. Let T be the standard simplex (see [DO] Fig 1.26b). $M_T = \{\pi, \pi/2, \theta = \arctan\sqrt{2} = \arccos\frac{1}{\sqrt{3}}\}$.

Since $\frac{1}{\pi}\arccos\frac{1}{\sqrt{n}}$ is irrational for $n = 3$ (see below), $\dim_{\mathbb{Q}} V(M_T) = 2 > 1 = \dim_{\mathbb{Q}} V(M_C)$. Therefore there exists a $\mathbb{Q}$-linear function $f : V(M_T) \to \mathbb{Q}$ such that $f(\pi) = 0$ and $f(\theta) = 23$. For this f, the corresponding Dehn invariant $D_f(T) = 3sf(\pi/2) + 3\cdot s\sqrt{2}\cdot f(\theta)$ is nonzero, but any Dehn invariant of the cube is 0, and hence by Dehn–Hadwiger theorem, T is not scissors congruent to a cube.

Exercise 6 (1.65)**.** Regular tetrahedron is not scissors congruent to a cube. (This is in [AZ§8], please do *not* look up the solution.)

Theorem 7 (Sydler 1.67)**.** *If P and Q are* not *scissors congruent, then there is some f such that the Dehn invariant $D_f(P) \neq D_f(Q)$.*

§4. Irrationality [AZ§6].

Theorem 8. *For odd integer $n \geq 3$,*

$$\frac{1}{\pi} \arccos \frac{1}{\sqrt{n}}$$

is irrational.

Proof. Recall addition formula

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta.$$

Summing yields

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta.$$

Let $\varphi_n = \arccos \frac{1}{\sqrt{n}}$, so $\cos\varphi_n = \frac{1}{\sqrt{n}}$. Substituting $\alpha = k\varphi_n$ and $\beta = \varphi_n$ yields

$$\cos(k+1)\varphi_n + \cos(k-1)\varphi_n = 2\cos\varphi_n \cos k\varphi_n$$

Claim 8.1. *For all integers $k \geq 0$,*

$$\cos k\varphi_n = \frac{A_k}{\sqrt{n}^k}$$

for some integer $A_k \in \mathbb{Z}$ such that $n \nmid A_k$.

Proof. Indeed, $A_0 = A_1 = 1$. By induction, we have

$$\begin{aligned} \cos(k+1)\varphi_n &= 2\cos\varphi_n \cos k\varphi_n - \cos(k-1)\varphi_n \\ &= 2\frac{1}{\sqrt{n}}\frac{A_k}{\sqrt{n}^k} - \frac{A_{k-1}}{\sqrt{n}^{k-1}} = \frac{2A_k - nA_{k-1}}{\sqrt{n}^{k+1}}, \end{aligned}$$

so

$$A_{k+1} = 2A_k - nA_{k-1}$$

is an integer. Moreover, if $n \mid A_{k+1}$ then $n \mid 2A_k$. But $n \geq 3$ is odd and $n \nmid A_k$, a contradiction. ☐

Suppose, towards a contradiction, that

$$\frac{1}{\pi} \arccos \frac{1}{\sqrt{n}} = \frac{p}{q}$$

with integers $p, q > 0$, then taking cosine of both sides of

$$q\varphi_n = p\pi$$

yields

$$\pm 1 = \cos p\pi = \cos q\varphi_n = \frac{A_q}{\sqrt{n}^q},$$

and hence $\sqrt{n}^q = \pm A_q$ is an integer. In fact, $q > 1$ as $0 < \arccos \frac{1}{\sqrt{n}} < \pi/2$. So $q \geq 2$ and $n \mid \sqrt{n}^q \mid A_q$, a contradiction. ☐