# Math 4990 Catalan Numbers

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Enumerative combinatorics is a field of mathematics that deals with counting the number of objects satisfying some combinatorial description. Catalan numbers count a wide variety of objects.

## 1. Monotonic paths

A **monotonic path** is a finite sequence of 'up' and 'right' steps of unit length.

**Problem 1.** Count the number of monotonic paths from $(0,0)$ to $(n,m)$.

**Problem 2.** Consider monotonic paths from $(0,0)$ to $(n,n)$ that cross above the diagonal, *i.e.*, those that enter the region $y > x$. Count the number of such paths by establishing a bijection with monotonic paths from $(0,0)$ to $(n-1,n+1)$.

**Problem 3.** Let $\mathcal{P}_n$ be the set of monotonic paths from $(0,0)$ to $(n,n)$ that do not cross above the diagonal, *i.e.*, those that stay in the region $y \leq x$. Count the number $|\mathcal{P}_n|$ of such paths.

**Problem 4.** Write the answer from the previous exercise as a fraction of a binomial coefficient. This is known as the *Catalan number* $C_n$.

## 2. Triangulations

Consider a convex $(n+2)$-gon with its vertices cyclically labelled 1 to $n+2$. Let $\mathcal{T}_{n+2}$ denote the set of its triangulations. Given a triangulation, define the following monotonic path. For each $v = 3, 4, \ldots, n+2$, in that order, take a 'right' step, followed by as many 'up' steps as the number of $u$, $1 \leq u \leq v$, such that $uv$ is a diagonal. End with an additional 'up' step.

**Problem 5.** Prove that this is a map $\mathcal{T}_{n+2} \to \mathcal{P}_n$.

**Problem 6.** Establish an inverse map $\mathcal{P}_n \to \mathcal{T}_{n+2}$. Conclude that the number of triangulations of a convex $(n+2)$-gon is $C_n$.

## 3. Ballot sequences

**Problem 7.** Let $\nearrow = (1,1)$ and $\searrow = (1,-1)$ be vectors in $\mathbb{Z}^2$. Let

$$S = \left\{ (v_1, v_2, \ldots, v_{2n}) \in \{\nearrow, \searrow\}^{2n} : \sum_{i=1}^{2n} v_i = (2n, 0) \right\}$$

be the collection of all sequences of length $2n$ with the same number of $\nearrow$ as $\searrow$. Count the number $|S|$ of such sequences.

**Problem 8.** Let $B$ be the sequences $(v_1, \ldots, v_{2n}) \in S$ such that each partial sum $s_t = \sum_{i=1}^{t} v_i$ has non-negative $y$-coordinate for $t \in [2n] = \{1, 2, \ldots, 2n\}$. These are known as *ballot sequences*—a way for $2n$ people to cast yes/no votes one at a time so that there are always at least as many "yes" as "no" votes, with a tie as the end result. Count the number $|B|$ of ballot sequences.

**Problem 9.** Prepend $v_0 = \nearrow$ in front of every sequence in both $S$ and $B$, and call the resulting sequences *augmented* and denote the sets $\overline{S}$ and $\overline{B}$, respectively. For each augmented sequence $(v_0, v_1, \ldots, v_{2n}) \in \overline{S}$, show that there is a unique $r \in [0, 2n] = \{0, 1, \ldots, 2n\}$ such that $(v_{0+r}, v_{1+r}, \ldots, v_{2n+r})$ is an augmented ballot sequence in $\overline{B}$, where the indices are read modulo $2n+1$. Call the map $f : \overline{S} \to \overline{B}$ that sends the augmented sequence to its corresponding augmented ballot sequence.

**Problem 10.** Show that $f$ is surjective and that the fibers under $f$ are all of the same size, and calculate this size. That is, calculate $\left|f^{-1}(b)\right|$ for $b \in B$.

**Problem 11.** Establish a linear relationship between $|B|$ and $|S|$.

**Problem 12.** Conclude a second proof of the expression of the Catalan number in **Problem 4** *without* using subtraction, but using division instead.