Math 4990 Problem Set 13

Due Tuesday, Dec 9, 2014 in class

Read [DO] Sections 7.2 and 7.3 carefully before doing the homework. In lieu of lecture, a synopsis of important points is provided below.

Optionally read Sections 7.1 and 7.4, *i.e.*, liberally peruse **pages 206–236** of [DO].

Assignment

[DO] Exercises 7.7, 7.9, 7.11, 7.15, and 7.16.

Make sure to supply enough details (especially for 7.7, do not just write down the answer, but explain a little).

Exercises 7.9 and 7.11 will be graded together. You may skip 7.9 if your proof of 7.11 proves (a stronger version of) 7.9 (without using 7.9).

Synopsis of Reading

§1. Robots and polygonal chains. For $\ell_i > 0$, let $A = [\ell_1, \dots, \ell_n]$ denote a **polygonal chain** of n links. A **configuration** C of a polygonal chain A is given by

$$C = (v_0, v_1, \dots, v_n)$$

where $v_i \in \mathbb{R}^2$, $v_0 = (0, 0)$ is at the origin, and $\ell_i = |v_i - v_{i-1}|$ is the Euclidean distance between v_{i-1} and v_i.

Given a configuration C, define θ_0 as the angle of the vector v_1, as measured anti-clockwise from the $+x$-axis (see Figure 7.9), and angles θ_i, $i = 1, \dots, n-1$, as the angle $v_{i+1}v_iv_{i-1}$. It is clear that for a given chain A, the angles $\Theta = (\theta_0, \theta_1, \dots, \theta_{n-1})$ completely determine C. Indeed, for example, $v_1 = (\ell_1 \cos\theta_1, \ell_1 \sin\theta_1)$, and the other coordinates can be written out explicitly as well.

Imagine the chain as a robotic arm and there are obstacles in its path (see Figure 7.11a). Some values of Θ will force the chain to intersect the obstacles. Figure 7.11b has the values of Θ that causes the chain to intersect the obstacles shaded yellow.

If we consider the chains in the abstract and allow self-intersections, the configuration space can be represented as a hyper-torus, as $\theta_i = 0$ and $\theta_i = 2\pi$ represents the same configuration. To make sure you understand this point, do **Exercise 7.7**.

If there are no obstacles, the possible locations of v_n form an annulus around the origin. Read the proof of Theorem 7.8 and look at the pretty pictures (Figures 7.12 and 13) to understand this. One sentence of the proof is left as **Exercise 7.9**. In **Exercise 7.11**, prove a stronger version of this theorem.

§2. Folding and intractable problems. NP-complete is a class of problems that are considered intractable, *i.e.*, we (probably)[1] cannot solve these problems "quickly." All this can be made rigorous (see *e.g.* Wikipedia) but we won't get into it here.

The Set Partition problem is known to be NP-complete.

The Ruler Folding problem can be used to solve Set Partition quickly. So if we can solve Ruler Folding quickly, we can then solve Set Partition quickly, (probably) a contradiction. So we (probably) cannot solve Ruler Folding quickly.

The reduction of using Ruler Folding to solve Set Partition is the content of Theorem 7.13, which you are encouraged to read and understand. Many NP-completeness proofs have this flavour—ingeniously using one problem to solve another problem. I find this kind of proofs very beautiful and quite pleasant to think about.[2]

§3. Unfolding is knot easy. Here we study what happens when the polygonal chains live in $\mathbb{R}^3$, *i.e.*, $C = (v_0, v_1, \ldots, v_n)$ with $v_i \in \mathbb{R}^3$, and the segments $v_i v_{i+1}$ are not allowed to intersect. Can we continuously deform the polygonal chain, *i.e.*, move in the configuration space, and make the chain lie in any configuration?

In other words, is there a configuration that is so tangled that it cannot be made straight? Theorem 7.14 proves that this is possible. It uses the fact that the trefoil knot (see *e.g.* Wikipedia) cannot be unknotted, a result we will take for granted as proving such is beyond the scope of this course.

Do **Exercise 7.15** to show that the theorem is best possible in the sense that any polygonal chain with fewer links can be made straight. Also, do **Exercise 7.16**.

Announcements

This assignment is due (along with the previous one) in class on Dec 9. However, if you feel like you do not understand the material enough to do the homework due to not having lecture, please email me with questions and we'll work something out, which may possibly include an extension on the homework so I can explain concepts to you in person.

[1] The famous unsolved problem $P = NP$ asks whether we can solve any of these problems quickly. It is widely believed that $P \neq NP$ and, hence, NP-complete problems *cannot* be solved quickly. However, since this is not yet proven, we will write "(probably)" as a shorthand for "assuming $P \neq NP$" for this discussion. If you solve $P = NP$, you will be instantly famous and also win a million dollars.

[2] Full disclosure: I have several published results on NP-completeness.