Math 4990 Problem Set 8

Due Tuesday, Oct 28, 2014 in class

Please try to do the assignment without consulting the Internet or books.

If you really must use other resources, cite your source and write your solution in your own words without copying. For example, in the last homework set, several people used undefined terminology that I recognize from various sources.

Let T_n be the number of domino tilings of a $2 \times n$ region. Recall the *linear recurrence relation*

$$T_n = T_{n-1} + T_{n-2} \qquad (*)$$

for $n \geq 3$. Using (*) for $n = 2$ suggests that it is sensible to define $T_0 = 1$. For the rest of the problem set, it is advisable to **NOT** use (*). That is, each time you see T_n, interpret it as "number of tilings" as opposed to the actual number.

Problem 1. Prove that $T_{a+b} = T_a T_b + T_{a-1} T_{b-1}$.
[**Hint:** Partition all domino tilings of a $2 \times (a+b)$ region into two types, one type having $T_a T_b$ tilings and the other $T_{a-1} T_{b-1}$.]

Problem 2. Prove that $\binom{n}{1} T_0 + \binom{n}{2} T_1 + \cdots + \binom{n}{n} T_{n-1} = T_{2n-1}$.
[**Hint:** Associate a number i to each domino tiling of a $2 \times (2n-1)$ region, $1 \leq i \leq n$, such that there are $\binom{n}{i} T_{i-1}$ tilings associated with i.]

Problem 3. Let $f(n)$ denote the number of domino tilings of a $3 \times n$ region, and let $g(n)$ denote the number of domino tilings of the same region but with one corner square removed.

(1) Calculate $f(n)$ and $g(n)$ from their definitions for $1 \leq n \leq 4$.
(2) Write $f(n)$ in terms of $f(a)$ and $g(b)$ for some $a, b < n$.
[**Hint:** Follow the idea that allowed us to find (*).]
(3) Similarly, write a recurrence relation for $g(n)$ in terms of f and g.
(4) Using the recurrence relations, define $f(0)$ and $g(0)$ sensibly.
(5) Obtain a linear recurrence relation of f alone by eliminating g.
(6) Let $h(n) = f(2n)$ and calculate $h(n)$ for $0 \leq n \leq 9$.
[**Hint:** One way to do this is by using Wolfram Alpha. For example, to get the first ten Fibonacci numbers, go to `http://www.wolframalpha.com`, enter

```
f(n)=f(n-1)+f(n-2), f(0)=0, f(1)=1
```

and click "more" on the result page.]
(7) What is the number of domino tilings of a 3×40 region?
[**Hint:** One way to do this is by using the On-Line Encyclopedia of Integer Sequences. Go to `http://oeis.org` and search with the first few terms of the sequence.]