Math 4707 Midterm 2

You may use books, notes, and calculators on this exam. Calculators will not be necessary. Please refrain from using other electronic devices such as laptops or cell phones. Do your work individually without collaboration. You have the full class period for the exam, and may leave early after turning in your work.

Assume that all graphs are simple, unless otherwise stated.

There are 5 problems, each worth 7 points. Please write your solutions carefully, showing all your work and justifying your steps rigorously. If you use tools from Chapter 2 such as induction, inclusion-exclusion, or pigeonholes, please state so explicitly.

Problem 1. Prove that a graph G (with at least two vertices) has two vertices with the same degree.

Solution. Let G be a graph on $n \geq 2$ vertices. Consider $n-1$ boxes labelled $1, 2, \ldots, n-1$. Put isolated vertices into the box labelled $n-1$, and put other vertices into boxes according to their degrees. Note that vertices of degrees 0 and $n-1$ cannot coexist in a graph on n vertices. Therefore all vertices in the box labelled $n-1$ have the same degree. It remains to show that there are two vertices in the same box, which is guaranteed by the pigeonhole principle, as there are n vertices and $n-1$ boxes. □

Problem 2. Let G be a graph and $\overline{G}$ its complement. Prove or disprove the following statements.

(a) If G is connected, then $\overline{G}$ is connected.
(b) If G is connected, then $\overline{G}$ is disconnected.
(c) If G is disconnected, then $\overline{G}$ is connected.
(d) If G is disconnected, then $\overline{G}$ is disconnected.

Solution. Note that cycles of length 3 and 5 are counter-examples to (a) and (b), respectively. The empty graph on 23 vertices is a counter-example to (d).

Only (c) is true. Indeed, suppose G is disconnected. For two vertices $x, y \in V(G)$ in the same connected component, pick $z \in V(G)$ in a different connected component, and note that xzy is a path in $\overline{G}$. Otherwise, x and y are not joined, so they are joined by an edge in $\overline{G}$. This means $\overline{G}$ is connected. □

Problem 3. Recall that there are n^{n-2} (labelled) trees on $[n] = \{1, 2, 3, \ldots, n\}$.

(a) Determine the number of trees on $[n]$ where vertex i is a leaf for every odd integer i.
(b) Determine the number of trees on $[n]$ where the degrees of the vertices 1 and n sum up to n.

Try to write *closed-form* answers, *i.e.*, without the use of summations or variables other than n.

Solution. (a): Let $k = \lceil n/2 \rceil$ be the number of odd integers between 1 and n, inclusive. Given such a tree, remove the odd vertices to get a tree on $n-k$ vertices. There are $(n-k)^{n-k-2}$ of these. Now add the k vertices back. Each one has $n-k$ potential places to go. Therefore there are $(n-k)^{n-k-2}(n-k)^k = (n-k)^{n-2} = (n - \lceil n/2 \rceil)^{n-2} = (\lfloor n/2 \rfloor)^{n-2}$ possible trees.

(b): The numbers of edges incident to 1 and to n sum up to n. Since there are only $n-1$ edges, (at least) one edge must be counted twice. Therefore 1 and n are joined, and all other vertices are joined to precisely one of the two. Hence there are 2^{n-2} such trees. □

Problem 4. Complete the statement of the theorem below by choosing integers a and b, and then prove the theorem.

Theorem. *Let G be a graph on 100 vertices and 30 edges. If G has c connected components, then*

$$a \leq c \leq b. \tag{*}$$

Moreover, the bounds in (*) are best possible.

(a) Pick a number a for the lower bound of the number c of components in (*). Prove the assertion that $a \leq c$, and then give an example of G where $c = a$ to prove the "best possible" assertion.
(b) Similarly, pick a number b for the upper bound of c in (*). Prove the assertion that $c \leq b$ and give an example where $c = b$.

[Hint: Part (b) might be substantially harder than part (a).]

Solution. (a): $a = 70$. Let us start with an empty graph on 100 vertices, which has 100 connected components. When an edge is added, the number of components could decrease by at most one. So after adding 30 edges, the number of components is at least $100 - 30 = 70$. The bound is achieved by any acyclic graph, *e.g.*, a path of length 30 with 69 isolated vertices.

(b): $b = 92$. Suppose there are $c > 92$ components. Then we may pick c vertices in different components, and add $100 - c$ vertices to existing components, one at a time. When adding a vertex to a component of size t, we can add up to t edges. Therefore this graph has at most $1 + 2 + 3 + \ldots + (100 - c)$ edges (less if we do not add vertices to the same component). Since $100 - c < 8$, we added at most $1 + 2 + \ldots + 7 = 28$ edges, a contradiction. The bound is achieved by joining a vertex to two vertices of a K_8 and adding 91 isolated vertices. ☐

Problem 5. Let $G = (V, E)$ be a graph and $w : E \to \mathbb{R}_+$ a weight function on its edges. Given a matching $M \subseteq E(G)$, consider the weight $w(M) = \sum_{e \in M} w(e)$, which is the sum of the weights of the edges in the matching. Let $m(G)$ denote the *maximum* possible value of $w(M)$ when M ranges over all matchings of G.

Consider the following greedy algorithm:

(1) Start with an empty matching $M = \varnothing$.
(2) Add to M an edge e of maximum weight from $E(G) \backslash M$ (edges not yet selected), such that M (with e added) is a matching.
(3) Repeat step (2) until such an edge e cannot be found, and output the matching M.

Answer the following questions:

(a) (**5 points**) Prove that the greedy algorithm always finds a matching M such that $w(M) \geq \frac{1}{2} m(G)$.
(b) (**2 points**) Suppose that G is a bipartite graph that admits a perfect matching. Let $w(e) = 1$ for every edge $e \in E(G)$ and run the greedy algorithm. Show that the greedy algorithm outputs a matching that covers at least half of the vertices.

[Hint: You can get **2 points** for proving (b) by using (a), even if you do not prove (a). On the other hand, if you skip (a) and prove (b) *without using* (a), you can get up to **4 points**.]

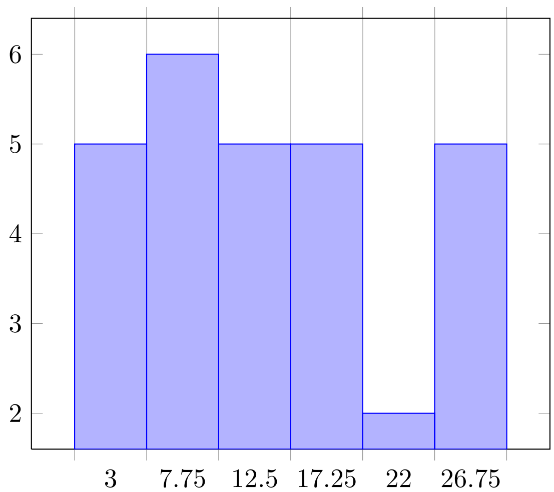
Solution. (a): Let M be an optimal matching, with $w(M) = m(G)$. And suppose $\{e_1, e_2, \ldots, e_k\}$ is an output of the greedy algorithm, where e_i is the edge chosen in round i. Consider an edge $f \in M$. If no edge e_i intersects f, then f can be added as e_{k+1} to form a better (greedy) matching, a contradiction. Thus we may let i_f be minimal such that e_{i_f} intersects f. Note that at round i_f, f could have been chosen as e_{i_f}. Therefore we conclude that $w(f) \leq w(e_{i_f})$. Summing these inequalities over all edges $f \in M$ yields

$$m(G) = w(M) = \sum_{f \in M} w(f) \leq \sum_{f \in M} w(e_{i_f}) \leq 2 \sum_{i=1}^{k} w(e_i),$$

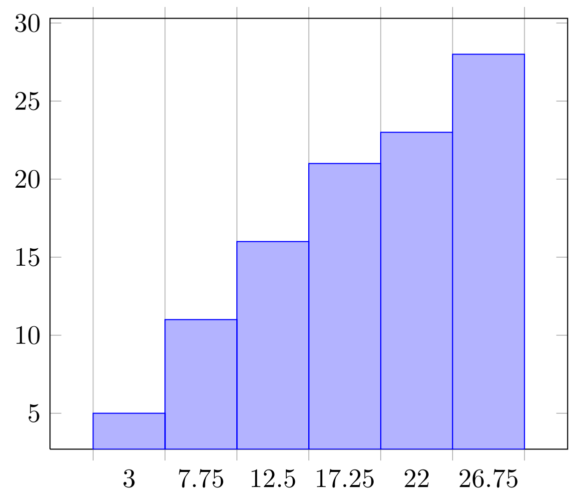
where the last inequality is because each edge e_i can occur as e_{i_f} at most twice (once for each of its ends). This means the greedy matching $\{e_1, \ldots, e_k\}$ is at least half as good as the optimal one, as desired.

(b): If $w(e) = 1$ for every edge $e \in E(G)$, then the weight is simply the number of edges. Suppose G is a bipartite graph with n vertices on each side. A perfect matching has weight n, so $m(G) = n$. By part (a), the greedy algorithm finds a matching with weight at least $n/2$, which covers at least n vertices, as desired. ☐

Problem	Mean	Stdev
Problem 1 (7 points)	2.39	3.31
Problem 2 (7 points)	5.82	1.98
Problem 3 (7 points)	3.00	2.52
Problem 4 (7 points)	3.80	2.29
Problem 5 (7 points)	1.21	1.75
$\sum$ (35 points total)	16.23	8.33



(A) Histogram of scores.



(B) Cumulative histogram of scores.