Math 4707 Midterm 1 Solutions

You may use books, notes, and calculators on this exam. Calculators will not be necessary. Please refrain from using other electronic devices such as laptops or cell phones. Do your work individually without collaboration. You have the full class period for the exam, and may leave early after turning in your work.

Problems 1 through 4 are worth 7 points each. Please write your solutions carefully, showing all your work and justifying your steps rigorously. If you use tools from Chapter 2 such as induction, inclusion-exclusion, or pigeonholes, please state so explicitly.

For problems 5 through 7, you do **not** need to show your work. You may simply write down your final answer in terms of binomial coefficients, factorials, and numbers. You do **not** need to simplify your answers algebraically. You will be given 1 point for each correct answer, for a total of 15 points.

Problem 1. The sequence a_n is given by $a_0 = 5$, $a_1 = 7$, and $a_n = 4a_{n-1} + 5a_{n-2}$ for $n \geq 2$. Find a formula for a_n for all natural number $n \in \mathbb{N}$.

Solution. The associated characteristic polynomial is $x^2-4x-5 = (x-5)(x+1)$, with roots $x = 5$ and $x = -1$. As such, both 5^n and $(-1)^n$ are potential solutions to the recurrence relation. Write $a_n = b \cdot 5^n + c \cdot (-1)^n$. Substituting $n = 0$ and $n = 1$ gives $5 = a_0 = b + c$ and $7 = a_1 = 5b - c$, respectively. Thus $b = 2$ and $c = 3$, yielding the formula $a_n = 2 \cdot 5^n + 3 \cdot (-1)^n$. □

Problem 2. Prove that $n^3 - n$ is divisible by 6 for all natural numbers $n \in \mathbb{N}$.

Solution. This is clearly true for $n = 1$. We proceed by induction on n. Suppose $n^3 - n$ is divisible by 6, then $n^3 - n = 6k$ for some integer $k \in \mathbb{Z}$. Then $(n + 1)^3 - (n + 1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3n^2 + 3n = 6k + 3n(n + 1)$. Since either n or $n + 1$ is even, $3n(n + 1)$ is divisible by 6, thus so is $(n + 1)^3 - (n + 1)$, completing the induction step. □

Problem 3. How many ways can you distribute n identical balls to g girls and b boys such that *each* girl gets at least as many balls as *all* the boys combined? [You may use summation notation in your answer.]

Solution. Let i be the number of balls we distribute to the boys in total. Give each girl i balls first, and then distribute the remaining $n - gi - i$ balls among the girls without restriction. There are $\binom{i+b-1}{b-1}$ ways to distribute i balls to the boys, and $\binom{n-gi-i+g-1}{g-1}$ ways to distribute the remainder to the girls. Summing over all possibilities of i, we get the final answer

$$\sum_{i=0}^{n} \binom{i+b-1}{b-1}\binom{n-gi-i+g-1}{g-1}.$$

□

Problem 4. You have 7 blue pens, 4 red pens, 4 purple pens, and 2 pink pens in your pocket. You take some pens out without looking.

(a) How many pens do you have to take out to make sure you have 2 pens of the same colour?
(b) How many pens do you have to take out to make sure you have 2 pens with different colours?

Solution. (a): Obviously 4 pens is not enough, since they could all be of different colours. By the pigeon hole principle, if there are 5 pens (pigeons), with 4 colour possibilities (holes), at least 2 pens will be of the same colour.

(b): Obviously 7 pens is not enough, as they could all be blue. However, with 8 pens, there must be pens with different colours as there are no 8 pens all of the same colour. □

Problem 5. Suppose there are 100 people. Count the number of different ways to do each of the following:

(a) Form a committee with 5 members.
(b) Form a committee with 5 members total, where 2 of whom are designated as "co-chairs."
(c) Partition into 3 groups of sizes 50, 30, and 20, respectively, where no one is in multiple groups.
(d) Partition into 4 equal-sized groups, where everyone is in precisely one group.

Solution.

(a): $\binom{100}{5}$.

(b): $\binom{100}{2}\binom{98}{3} = \binom{100}{5}\binom{5}{2}$.

(c): $\binom{100}{50,30,20} = \binom{100}{50}\binom{50}{30}\binom{20}{20} = \frac{100!}{50!30!20!}$.

(d): $\binom{100}{25,25,25,25}/4! = \frac{100!}{4!25!25!25!25!}$. The factor of 4! in the denominator is to compensate for the fact that we cannot tell the four groups apart. □

Problem 6. Let $n \in \mathbb{N}$ be a natural number and consider a $3n$-by-$3n$ grid. Let X denote the set of shortest (monotonic) paths along the grid lines from $A = (0,0)$ to $B = (3n, 3n)$. Find the number of such paths given certain restrictions. [Your answer should be in terms of n, and can have multiple binomial coefficients.]

(a) No restrictions; in other words, find $|X|$.
(b) Visit $C = (n, n)$ and $E = (2n, 2n)$.
(c) Visit C or E (or both).
(d) Visit $F = (n, 2n)$ or $D = (2n, n)$.
(e) Avoid C, D, E, and F.

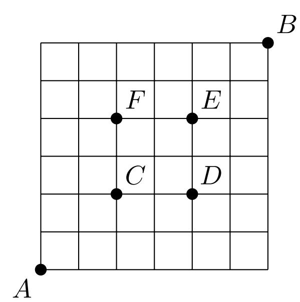


Figure 6: The case for $n = 2$.

Solution.

(a): $\binom{6n}{3n}$.

(b): $\binom{2n}{n}^3$.

(c): $2\binom{2n}{n}\binom{4n}{2n} - \binom{2n}{n}^3$, by principle of inclusion-exclusion.

(d): $2\binom{3n}{n}^2$; note that it is impossible to visit both F and D.

(e): $\binom{6n}{3n} - 2\binom{2n}{n}\binom{4n}{2n} - 2\binom{3n}{n}^2 + \binom{2n}{n}^3 + 4\binom{2n}{n}\binom{3n}{n} - 2\binom{2n}{n}^2$, by principle of inclusion-exclusion. □

Problem 7. Suppose you have a small garden divided into a 2-by-3 grid (see figure below). You have 6 flowers and plan to plant each one in a different square in the grid. Count the number of different floral arrangements under the following circumstances:

(a) You have 6 different kinds of flowers.
(b) You have 4 chrysanthemums that you consider the same, 1 rose, and 1 dandelion.
(c) You have 4 chrysanthemums that you consider the same and 2 roses that you consider the same.

Now suppose you are putting 6 cookies in a 2-by-3 box, one in each compartment (see the same figure below). The difference between a box and a garden is that you can't tell if a box is turned around. [Do not flip the box upside-down, for the cookies will fall out and make a mess.] Count the number of different snack arrangements under the following circumstances:

(d) You have 6 different kinds of cookies.
(e) You have 4 chocolate-chip cookies that you consider the same, 1 raisin cookie, and 1 dandelion-flavoured cookie.
(f) You have 4 chocolate-chip cookies that you consider the same and 2 raisin cookies that you consider the same.

Figure 7: A garden divided into a 2-by-3 grid. Incidentally, also a 2-by-3 box with 6 compartments.

Solution.

(a): $6! = 720$.

(b): $6 \cdot 5 = 30$. Plant the rose, then plant the dandelion.

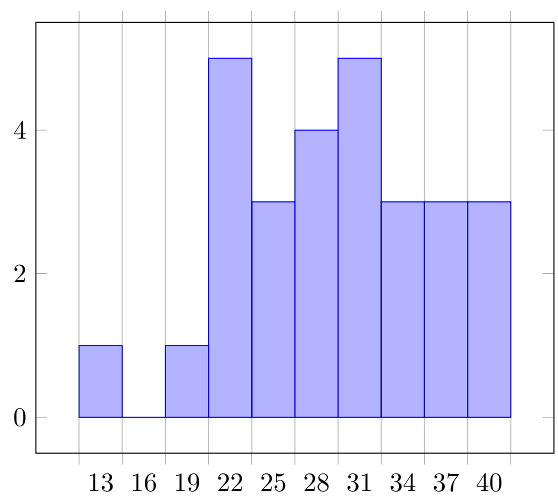
(c): $\binom{6}{2} = 15$. Plant the roses first.

(d): $6!/2 = 360$. Divide by 2 to account for rotation.

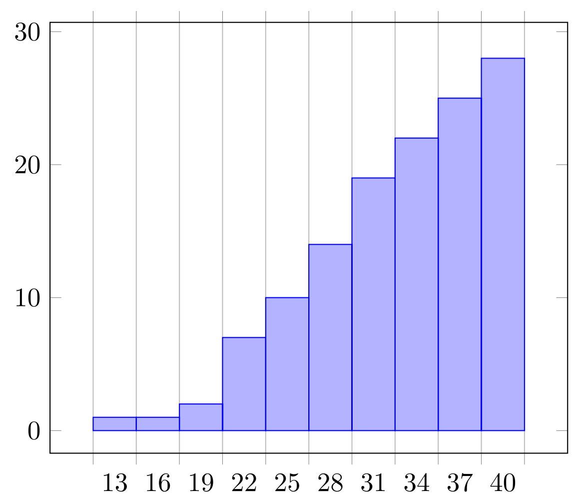
(e): $3 \cdot 5 = 15$. Put raisin in the top row, then put dandelion anywhere else.

(f): $3+(15-3)/2 = 9$. Three of the 15 from part (c) have rotational symmetry. The others are paired. $\square$

Problem	Mean	Stdev
Problem 1 (7 points)	5.93	2.02
Problem 2 (7 points)	5.54	2.63
Problem 3 (7 points)	2.00	2.82
Problem 4 (7 points)	6.46	1.64
Problem 5 (4 points)	2.71	0.81
Problem 6 (5 points)	3.29	1.38
Problem 7 (6 points)	4.32	1.02
$\sum$ (43 points total)	30.25	7.17



(a) Histogram of scores.



(b) Cumulative histogram of scores.