Math 1271-040 Midterm Exam 3 Solutions

Problem 1. Evaluate the following indefinite integrals.

(a) [**Exercise 5.5.18**]

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Solution. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$. By the Substitution Rule, we get

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin u\,du = 2(-\cos u) + C = -2\cos\sqrt{x} + C,$$

where C is an arbitrary constant. □

(b) [**Exercise 5.5.45**]

$$\int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

Solution. Let $u = 1 - x^2$. Then $du = -2x\,dx$. We get

$$\begin{aligned}
\int \frac{1-x}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{-x}{\sqrt{1-x^2}}\,dx \\
&= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{1}{2\sqrt{u}}\,du \\
&= \arcsin x + \sqrt{u} + C \\
&= \arcsin x + \sqrt{1-x^2} + C,
\end{aligned}$$

where C is an arbitrary constant. □

Problem 2. [**Exercise 5.4.59**] A particle moves along a line so that its velocity $v(t)$ at time t is given by

$$v(t) = \frac{3t-6}{\sqrt{t}}.$$

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

Solution. Let $s(t)$ denote the position of the particle at time t. We know that $s'(t) = v(t)$ and seek $s(4) - s(1)$. By the Net Change Theorem, we get

$$\begin{aligned} s(4) - s(1) &= \int_1^4 s'(t)\,dt = \int_1^4 v(t)\,dt = \int_1^4 \left(3t^{1/2} - 6t^{-1/2}\right)\,dt \\ &= \left.\frac{3t^{3/2}}{3/2} - \frac{6t^{1/2}}{1/2}\right]_1^4 = 2t^{3/2} - 12\sqrt{t}\Big]_1^4 = 2. \end{aligned}$$

□

(b) Find the distance travelled during the same time period $1 \le t \le 4$.

Solution. Here we integrate $|v(t)|$ instead of $v(t)$. Note that $v(t) \le 0$ for $1 \le t \le 2$ and $v(t) \ge 0$ for $2 \le t \le 4$. As such, by the properties of the definite integral, we get

$$\begin{aligned} \int_1^4 |v(t)|\,dt &= \int_1^2 -v(t)\,dt + \int_2^4 v(t)\,dt \\ &= -2t^{3/2} + 12\sqrt{t}\Big]_1^2 + 2t^{3/2} - 12\sqrt{t}\Big]_2^4 = 16\sqrt{2} - 18. \end{aligned}$$

□

Problem 3. [**Exercise 6.1.20**] Find the area of the region enclosed by the curves $x = y^4$, $y = \sqrt{2-x}$, and $y = 0$. [**Hint:** Sketch the region.]

Solution. Simultaneously solve $x = y^4$ and $y = \sqrt{2-x}$ yields $x = (2-x)^2$ or $0 = x^2 - 5x + 4 = (x-4)(x-1)$. Since $x = 4$ is not in the domain of $\sqrt{2-x}$, we know that the unique intersection point is at $(x, y) = (1, 1)$. (See sketch below.) It is easier to integrate with respect to y. We rewrite $y = \sqrt{2-x}$ as $y^2 = 2 - x$ or $x = 2 - y^2$. Integrating, we get

$$\int_0^1 (2 - y^2) - (y^4)\, dy = 2y - \frac{y^3}{3} - \frac{y^5}{5}\Big]_0^1 = 2 - \frac{1}{3} - \frac{1}{5} = \frac{22}{15},$$

which is the desired area. □

Solution. Alternatively, we can integrate with respect to x. We rewrite $x = y^4$ as $y = x^{1/4}$. The curve is given by

$$f(x) = \begin{cases} x^{1/4} & \text{for } 0 \le x \le 1 \\ \sqrt{2-x} & \text{for } 1 \le x \le 2. \end{cases}$$

The area is therefore

$$\begin{aligned} \int_0^2 f(x)\, dx &= \int_0^1 x^{1/4}\, dx + \int_1^2 \sqrt{2-x}\, dx \\ &= \left[\frac{4}{5}x^{5/4}\right]_0^1 + \left[-\frac{2}{3}(2-x)^{3/2}\right]_1^2 \\ &= \frac{4}{5} + \frac{2}{3} = \frac{22}{15}, \end{aligned}$$

which is, of course, the same answer as above. □

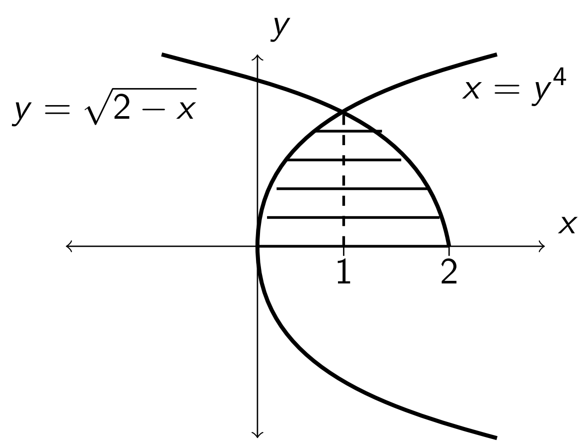


FIGURE 3. Sketch of the region for Problem 3, courtesy of G. Jaramillo.

Problem 4. Evaluate the following expressions involving the definite integral.

(a) [**Exercise 5.5.60**]

$$\int_0^1 xe^{-x^2}\,dx$$

Solution. Let $u = g(x) = -x^2$. Then $du = -2x\,dx$. Using the Substitution Rule for definite integrals, we get

$$\int_0^1 xe^{-x^2}\,dx = \int_{g(0)}^{g(1)} -\tfrac{1}{2}e^u\,du = -\tfrac{1}{2}e^u\Big]_0^{-1} = \tfrac{1}{2}(1-1/e).$$

□

(b)

$$\int_{-23}^{23} \left(1 + x^2\sin x + x^4\sin x + x^6\sin x\right)\,dx$$

Solution. Note that $f(x) = x^2\sin x + x^4\sin x + x^6\sin x = (x^2+x^4+x^6)\sin x$ is odd. Indeed $f(-x) = \left((-x)^2 + (-x)^2 + (-x)^6\right)\sin(-x) = (x^2+x^4+x^6)(-\sin x) = -f(x)$. As such, $\displaystyle\int_{-23}^{23} f(x)\,dx = 0$ by symmetry. This means that

$$\int_{-23}^{23} \left(1 + x^2\sin x + x^4\sin x + x^6\sin x\right)\,dx = \int_{-23}^{23} 1\,dx = x\Big]_{-23}^{23} = 46.$$

□

(c)

$$\frac{d}{dx}\int_{1-2x}^{x^2-1} \sin\sqrt{t}\,dt$$

Solution. Let $A(x) = \displaystyle\int_0^x \sin\sqrt{t}\,dt$. By FTC1, we have that

$$A'(x) = \sin\sqrt{x}.$$

Now, by the properties of the definite integral, we have that

$$\begin{aligned} F(x) := \int_{1-2x}^{x^2-1} \sin\sqrt{t}\,dt &= \int_0^{x^2-1} \sin\sqrt{t}\,dt - \int_0^{1-2x} \sin\sqrt{t}\,dt \\ &= A(x^2-1) - A(1-2x) \end{aligned}$$

By the Chain Rule, we have

$$\begin{aligned} F'(x) &= A'(x^2-1)\cdot(2x) - A'(1-2x)\cdot(-2) \\ &= 2x\sin\sqrt{x^2-1} + 2\sin\sqrt{1-2x}, \end{aligned}$$

defined when $x^2 - 1 \geq 0$ and $1 - 2x \geq 0$, *viz.*, $x \leq -1$. □

Problem 5.

(a) Suppose $f(x)$ is continuous on the interval $[a, b]$. Write down a definition of the definite integral

$$\int_a^b f(x)\,dx$$

using a limit of Riemann sums. You may use right endpoints.

Solution. Let $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$. The definite integral is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x.$$

□

(b) Use the definition above to evaluate the integral

$$\int_1^3 (x-5)\;dx.$$

Do not use FTC2 or any other method. These summation formulae may be helpful:

$$\sum_{i=1}^n 1 = n, \qquad \sum_{i=1}^n i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Solution. Let $f(x) = x - 5$. Following the definition above, we get $\Delta x = \dfrac{2}{n}$ and $x_i = 1 + \dfrac{2i}{n}$. So

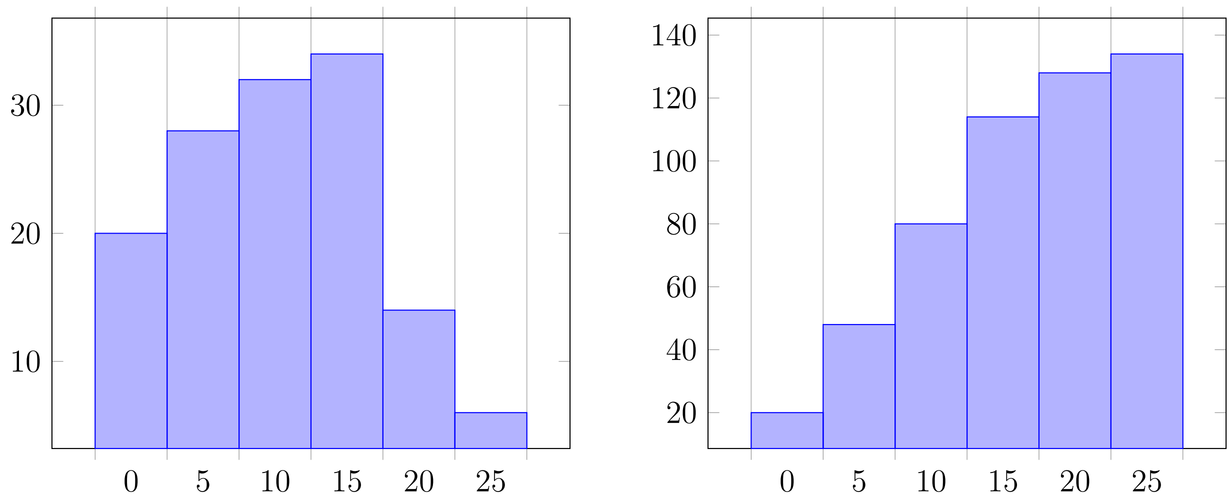
$$\begin{aligned}
\int_1^3 (x-5)\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(1 + \frac{2i}{n} - 5\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{4}{n^2}i - \frac{8}{n}\right) \\
&= \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^n i - \frac{8}{n}\sum_{i=1}^n 1 \\
&= \lim_{n\to\infty} \frac{4}{n^2}\frac{n(n+1)}{2} - \frac{8}{n}n \\
&= \lim_{n\to\infty} 2\left(1 + \frac{1}{n}\right) - 8 \\
&= 2\cdot(1+0) - 8 = -6
\end{aligned}$$

□

Problem	Mean	Stdev
Problem 1 (6 points)	2.15	1.90
Problem 2 (6 points)	2.32	1.67
Problem 3 (6 points)	2.65	2.12
Problem 4 (6 points)	1.90	1.48
Problem 5 (6 points)	3.29	1.90
$\sum$ (30 points total)	12.34	6.63

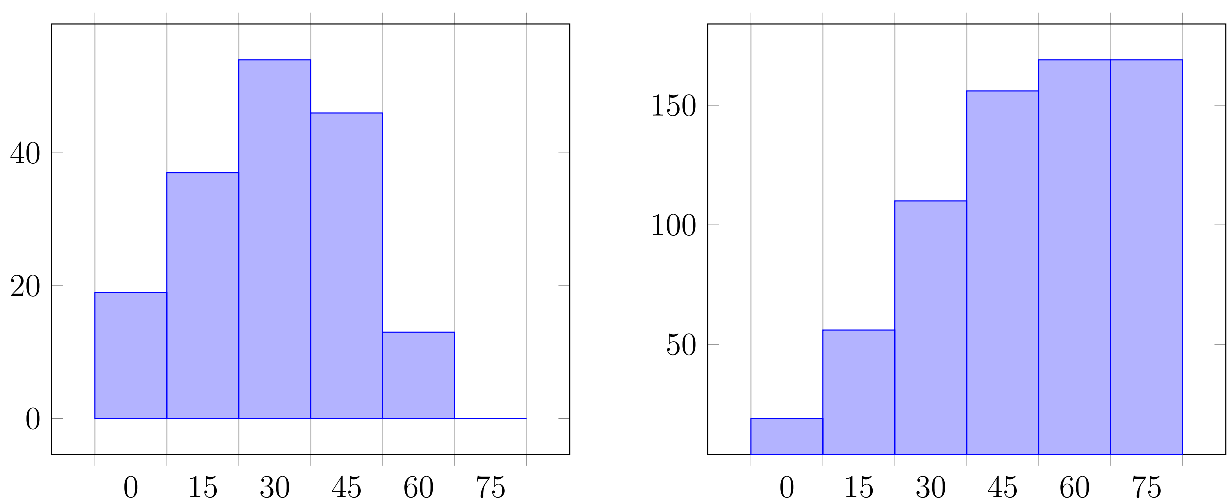
Below are score intervals, grade range estimates, and numbers of students scoring in each interval. Please note that midterm exams DO NOT have letter grades attached to them. These grade range estimates, based on historical distributions in MATH 1271 in past semesters, are meant to give you a sense of how you stand in the course to this point. As we've discussed, the final grade distribution is determined by the final exam score distribution, and your position in that distribution will be determined by all of your work in the course.

sum of Midterms 1, 2, and 3	grade range	number of students
$[51, 90]$	A	33
$[40, 50]$	B	39
$[25, 39]$	C	51
$[0, 24]$	in danger of failing	46



(A) Histogram of scores. (B) Cumulative histogram of scores.

Figure 4. Histograms for Midterm 3 scores.



(A) Histogram of scores. (B) Cumulative histogram of scores.

Figure 5. Histograms for Midterms 1, 2 and 3 scores combined.