Math 1271-040 Midterm Exam 2 Solutions

Problem 1. Calculate y'.
Answers can be in terms of both x and y; it is not necessary to simplify answers.

(a) [**Exercise 3.5.8**] $2x^3 + x^2y - xy^3 = 2$.

Solution. Use implicit differentiation. Differentiate both sides:

$$6x^2 + 2xy + x^2y' - y^3 - 3xy^2y' = 0.$$

Collate terms involving y' to one side

$$6x^2 + 2xy - y^3 = 3xy^2y' - x^2y'$$

and solve for y':

$$y' = \frac{6x^2 + 2xy - y^3}{3xy^2 - x^2}.$$

□

(b) $y = \dfrac{e^{-x}\cos^2 x}{(x^2+x+1)^5\sqrt{x-1}}$

Solution. Use logarithmic differentiation. Take logarithm of both sides:

$$\ln y = -x + 2\ln\cos x - 5\ln(x^2+x+1) - \tfrac{1}{2}\ln(x-1).$$

Differentiate both sides

$$\frac{y'}{y} = -1 + 2\frac{-\sin x}{\cos x} - 5\frac{2x+1}{x^2+x+1} - \frac{1}{2(x-1)}$$

and solve for y':

$$y' = y\left(-1 - 2\tan x - \frac{5(2x+1)}{x^2+x+1} - \frac{1}{2(x-1)}\right).$$

□

Problem 2. [**Chapter 3 review question 97**] The volume of a cube is increasing at a rate of 10 cubic meters per minute. How fast is the surface area increasing when the length of an edge is 5 meters?

Solution. Let s denote the length of an edge. The volume V of the cube is given by

$$V = s^3.$$

The surface area A of the cube is given by

$$A = 6s^2.$$

We are given that

$$\tfrac{dV}{dt} = 10$$

and want to know $\frac{dA}{dt}$ when $s = 5$. Write A in terms of V as

$$A = 6s^2 = 6V^{\frac{2}{3}}$$

and differentiate both sides to get

$$\tfrac{dA}{dt} = 6 \cdot \tfrac{2}{3} V^{-\frac{1}{3}} \tfrac{dV}{dt} = \tfrac{4}{s} \tfrac{dV}{dt}$$

using $V^{\frac{1}{3}} = s$. When $s = 5$, we get

$$\tfrac{dA}{dt}\Big|_{s=5} = \tfrac{4}{5} \cdot 10 = 8,$$

so the surface area is increasing at 8 square meters per minute. □

Problem 3. Evaluate the limits. Simplify answers but leave them exact (*e.g.*, do not use decimal approximations). Answers could be ∞, $-\infty$, or "does not exist."

(a) [**Exercise 4.4.61**] $\lim_{x\to\infty} x^{3/x}$

Solution. This is an indeterminate form of type ∞^0, so we may use l'Hôpital by taking the logarithm. Let

$$y = x^{3/x}.$$

Then

$$\ln y = \ln x^{3/x} = \tfrac{3}{x} \ln x.$$

Take limit of both sides:

$$\lim_{x\to\infty} \ln y = 3 \lim_{x\to\infty} \frac{\ln x}{x} = 3 \lim_{x\to\infty} \tfrac{1/x}{1} = 0,$$

where the middle equality follows from L'Hôpital's Rule. Finally:

$$\lim_{x\to\infty} x^{3/x} = \lim_{x\to\infty} y = e^{\left(\lim_{x\to\infty} \ln y\right)} = e^0 = 1,$$

where the middle equality follows from the continunity of e^x. □

(b) [[1]] $\lim_{x\to 0^+} \dfrac{\ln(1-x) - \sin x}{1 - \cos^2(x)}$

Solution. This is an indeterminate form of type $\frac{0}{0}$. L'Hôpital gives:

$$\lim_{x\to 0^+} \frac{\ln(1-x) - \sin x}{1-\cos^2(x)} = \lim_{x\to 0^+} \frac{\frac{-1}{1-x} - \cos x}{2\sin x \cos x}.$$

However, as

$$\lim_{x\to 0^+} \tfrac{-1}{1-x} - \cos x = -1 - 1 = -2$$

but

$$\lim_{x\to 0^+} 2 \sin x \cos x = 0,$$

the limit does not exist. In fact, since $2\sin x \cos x > 0$ for small $x > 0$, the limit is $-\infty$. □

[1] From the math competition scene from the film *Mean Girls* (2004).

(c) $\lim_{x\to 0} \dfrac{x\sin(x)\sin(2x)\sin(3x)}{x^3\sin(5x)}$

Solution. Using the fact that $\lim_{x\to 0}\frac{\sin nx}{n} = 1$ for any constant n, we immediately get that the limit is

$$\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin 2x}{2x}\cdot\frac{\sin 3x}{3x}\cdot\frac{5x}{\sin 5x}\cdot\frac{6}{5} = \frac{6}{5}.$$

□

Problem 4. [**Exercise 4.7.30**] A right circular cylinder is inscribed in a right circular cone with height h and base radius r. Find the largest possible volume of such a cylinder.

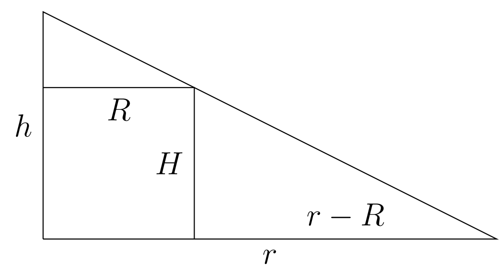


FIGURE 4. Cross section.

Solution. Consider an inscribed right circular cylinder with height H and base radius R (see Figure 4). By similar triangles, we get

$$\frac{r}{h} = \frac{r-R}{H}$$

so

$$H = \tfrac{r-R}{r}h.$$

Volume V of the cylinder is given by

$$V = \pi R^2 H = \pi R^2\cdot\tfrac{r-R}{r}h = \pi h R^2 - \tfrac{\pi h}{r}R^3.$$

Having written V in terms of a single variable R, we may maximize V by the Closed Interval Method for R in the interval $[0, r]$. Namely, set the derivative

$$\tfrac{dV}{dR} = 2\pi h R - \tfrac{3\pi h}{r}R^2$$

equal to 0 and solve for critical numbers:

$$\tfrac{dV}{dR} = 0 \qquad\Longrightarrow\qquad R = 0 \text{ or } R = \tfrac{2}{3}r.$$

Note that V is 0 when R is 0 or r, so the unique maximum is achieved when $R = \frac{2}{3}r$ and $H = \frac{r-R}{r}h = \frac{1}{3}h$. We finally get that

$$V = \pi R^2 H = \pi(\tfrac{2}{3}r)^2(\tfrac{1}{3}h) = \tfrac{4}{27}\pi r^2 h$$

is the largest possible volume. □

Problem 5. Estimate $\sqrt[4]{10008}$ in the following ways.
It is **not** necessary to simplify expressons (*e.g.*, sums, fractions) involving only numbers.

(a) Use a linear approximation (or differentials) to estimate.

Solution. Let
$$f(x) = x^{1/4}$$
and note that
$$f(10008) = \sqrt[4]{10008}$$
is what we want to approximate. Since $10008 \approx 10^4$ and $f(10^4) = 10$ is easy to calculate, we compute the linearization L of f at $x = 10^4$:
$$L(x) = f(10^4) + f'(10^4)(x - 10^4) = 10 + \frac{x - 10^4}{4000},$$
where
$$f'(x) = \tfrac{1}{4}x^{-3/4}.$$
Then we find the estimate
$$\sqrt[4]{10008} = f(10008) \approx L(10008) = 10 + \frac{8}{4000} = 10.002,$$
as desired. □

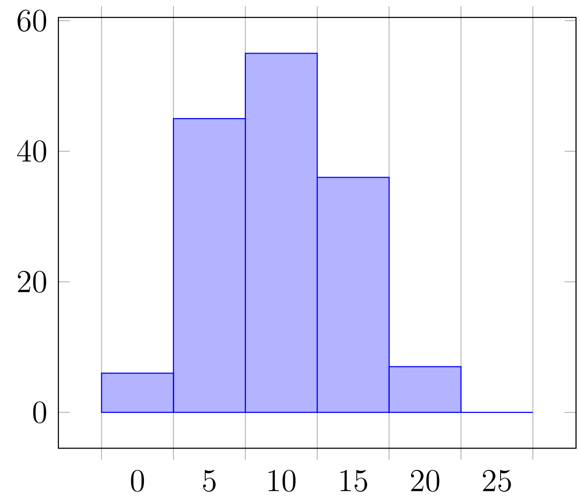
(b) Use Newton's Method to estimate: pick a sensible x_1 and calculate x_2.

Solution. Let
$$g(x) = x^4 - 10008$$
and note that $\sqrt[4]{10008}$ is the root of g around 10. We therefore pick $x_1 = 10$ and note that $g'(x) = 4x^3$. Newton's Method gives an estimate
$$x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = 10 - \frac{10000 - 10008}{4000} = 10 + \frac{8}{4000} = 10.002$$
of the root $\sqrt[4]{10008}$. □

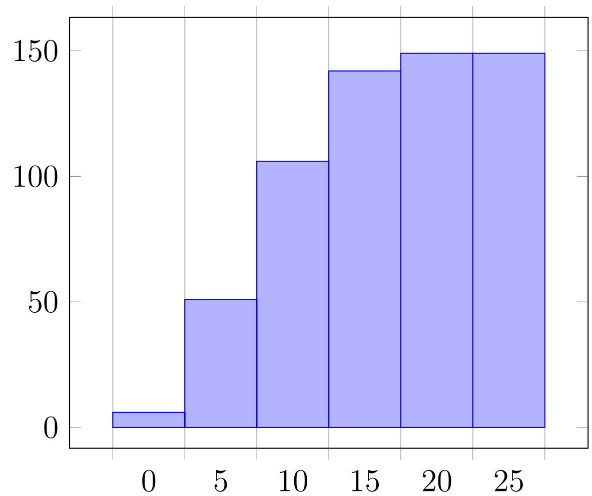
Problem	Mean	Stdev
Problem 1 (6 points)	3.07	1.62
Problem 2 (6 points)	3.74	1.41
Problem 3 (6 points)	2.68	1.10
Problem 4 (6 points)	0.13	0.45
Problem 5 (6 points)	2.20	1.61
$\sum$ (30 points total)	11.83	4.44

Below are score intervals, grade range estimates, and numbers of students scoring in each interval. Please note that midterm exams DO NOT have letter grades attached to them. These grade range estimates, based on historical distributions in MATH 1271 in past semesters, are meant to give you a sense of how you stand in the course to this point. As we've discussed, the final grade distribution is determined by the final exam score distribution, and your position in that distribution will be determined by all of your work in the course.

sum of Midterm 1 and Midterm 2	grade range	number of students
$[36, 60]$	A	33
$[30, 35]$	B	39
$[21, 29]$	C	51
$[0, 20]$	in danger of failing	46

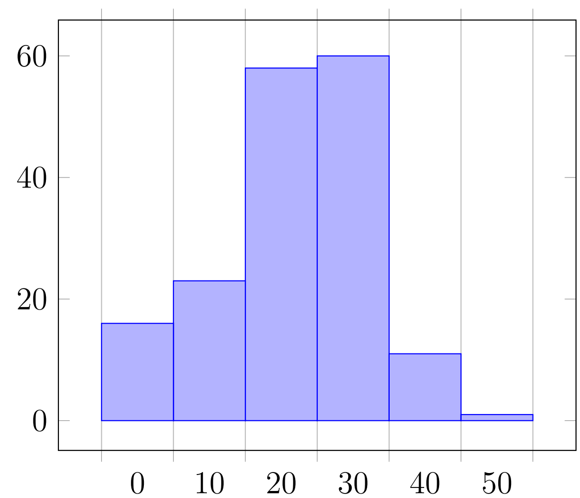


(A) Histogram of scores.

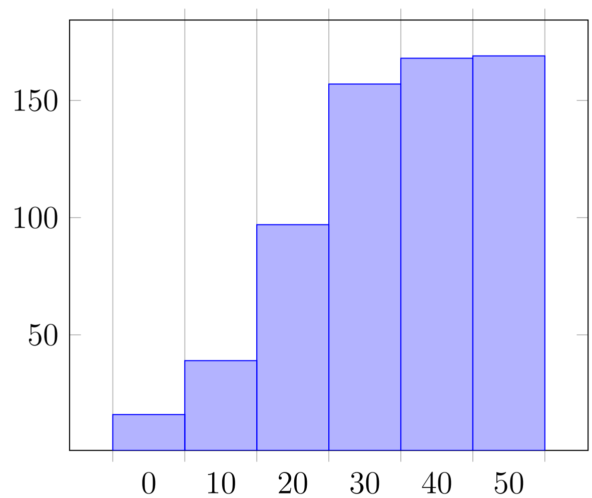


(B) Cumulative histogram of scores.

FIGURE 5. Histograms for Midterm 2 scores.



(A) Histogram of scores.



(B) Cumulative histogram of scores.

FIGURE 6. Histograms for Midterms 1 and 2 scores combined.