VOLUME OF A PRISM

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Theorem 1. *The volume of a prism with base area b and height h is given by $\frac{1}{3}bh$.*

Proof. Let us orient the prism with its vertex at the origin, and the height h running in the $+x$ direction. The volume of the solid is the integral of the cross sectional area, *viz.*,

$$V = \int_0^h A(x)\,dx,$$

where $A(x)$ is the area of the cross section at x.

Notice that $A(0) = 0$ and $A(h) = b$. To figure out the entire area function, recall that the area is proportional to x^2. (Try convincing yourself of this. For instance, if you take a rectangle and dilate it by a factor of k, then each side becomes k times as much, hence the area is k^2 as much. This is true also for a circle, or any other two-dimensional shape. Similarly, if you try this on a three dimensional shape, dilating in all dimensions by k will yield a k^3-fold increase in volume.) Thus we may conclude that

$$A(x) = b\left(\frac{x}{h}\right)^2.$$

Putting all this together, we get

$$V = \int_0^h \frac{bx^2}{h^2}\,dx = \frac{1}{3}bh,$$

as desired. □

In particular, the volume of the tetrahedron in Exercise 16.5.20 is

$$V = \tfrac{1}{3}(\tfrac{1}{2}ab)c = \tfrac{1}{6}abc,$$

where $\frac{1}{2}ab$ is the area of the triangular base.