MATH 32A DISCUSSION

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1. Directional Derivatives

1.1. **Basics.** If $f(x,y,z)$ is differentiable, we get the *gradient*

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

is a vector-valued function. Then the *directional derivative* of f in the direction of vector $\mathbf{u}$ is $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

1.2. **Exercise 15.6.4.** Find the directional derivative of $f(x,y) = x^2y^3 - y^4$ at $(2,1)$ in the direction $\theta = \pi/4$.

Solution. Here $\nabla f(x,y) = \left\langle 2xy^3, 3x^2y^2 - 4y^3\right\rangle$ so $\nabla f(2,1) = \langle 4,8\rangle$. Now $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle = \left\langle\sqrt{2}/2, \sqrt{2}/2\right\rangle$. Then $D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = 6\sqrt{2}$. ☐

1.3. **Exercise 15.6.16.** Find the directional derivative of $f(x,y,z) = \sqrt{xyz}$ at $(3,2,6)$ in the direction of the vector $\mathbf{v} = \langle -1,-2,2\rangle$.

Solution. Here $\nabla f(x,y,z) = \langle yz, xz, xy\rangle/2\sqrt{xyz}$, so $\nabla f(3,2,6) = \left\langle 1, \frac{3}{2}, \frac{1}{2}\right\rangle$. Now $\mathbf{u} = \mathbf{v}/\left|\mathbf{v}\right| = \left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right\rangle$. Then $D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = 1$. ☐

1.4. **Exercise 15.6.37b.** Assume that u and v are differentiable functions of x and y, show that $\nabla(uv) = u\nabla v + v\nabla u$.

Solution. Let us examine the first coordinate of both sides. On the left, we get $(uv)_x = uv_x + vu_x$, which is what we have on the left. Similarly for the second coordinate, so we are done. We also have rules such as $\nabla(au+bv)a\nabla u + b\nabla v$ for $a,b\in\mathbb{R}$, and $\nabla(\frac{u}{v}) = \frac{v\nabla u - u\nabla v}{v^2}$. ☐

1.5. **Exercise 15.6.42.** Find equations of the tangent plane and the normal line to the given surface

$$x - z = 4\arctan(yz)$$

at $(1+\pi, 1, 1)$.

Solution. Let $F(x,y,z) = 4\arctan(yz) - x + z$, then the surface is $F(x,y,z) = 0$, thus $\nabla F(1+\pi,1,1)$ gives a normal vector. Calculating, we get $\nabla F(x,y,z) = \left\langle -1, 4z/(1+y^2z^2), 1+4y/(1+y^2z^2)\right\rangle$, and $\nabla F(1+\pi,1,1) = \langle -1,2,3\rangle$. Tangent plane is given by $-(x-1-\pi) + 2(y-1) + 3(z-1) = 0$. Normal line is given by $\frac{x-1-\pi}{-1} = \frac{y-1}{2} = \frac{z-1}{3}$. ☐

1.6. **Exercise 15.6.50.** Find the equation of the tangent plane to the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ at (x_0, y_0, z_0).

Solution. Let $f(x,y,z) = x^2/^2 + y^2/b^2 - z^2/c^2$. Then a normal is given by $\nabla f = \langle 2x/a^2, 2y/b^2, -2z/c^2 \rangle$, so we have $\nabla f(x_0,y_0,z_0) = \langle 2x_0/a^2, 2y_0/b^2, -2z_0/c^2 \rangle$. An equation of the tangent plane is therefore $\frac{x_0}{a^2}(x-x_0) + \frac{y_0}{b^2}(y-y_0) - \frac{z_0}{c^2}(z-z_0) = 0$. Rearranging, we get $xx_0/a^2 + yy_0/b^2 - zz_0/c^2 = 1$, using the fact that (x_0,y_0,z_0) satisfies $f(x,y,z) = 1$. □

1.7. **Exercise 15.6.55,57,58.** These three problems are interesting and worth looking at.

1.8. **Exercise 15.6.59.** Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

Solution. We can take gradient of each and evaluate at the point $(-1,1,2)$, we get $\langle -2, 2, -1 \rangle$ and $\langle -8, 2, 4 \rangle$. Taking the cross product gives us a vector $\langle 10, 16, 12 \rangle$ that is the direction of the tangent line. The details are left as exercise. □

2. Local Extrema

2.1. **Basics.** The *critical points* of f are where $\nabla f = 0$ or ∇f is undefined. The local extrema only occur at critical points (but not all critical points are local extrema). Using the second derivative test, we have $D = f_{xx}f_{yy} - f_{xy}^2$; if $D < 0$ we get saddle point, if $D > 0$, then we get local min if $f_{xx} > 0$, and local max if $f_{xx} < 0$. In other cases, we don't know, and it could be anything.

2.2. **Exercise 15.7.11.** Find the local maximum and minimum values and saddle point(s) of the function $f(x,y) = x^3 - 12xy + 8y^3$.

Solution. Following the recipe, we calculate $\nabla f = \langle 3x^2 - 12y, -12x + 24y^2 \rangle$ and set it equal to $\langle 0, 0 \rangle$. So $x = 2y^2$ and $12y^4 - 12y = 0$, yielding $y = 0$ or $y = 1$, with $x = 0$ and $x = 2$, respectively. At $(0,0)$, $D(0,0) < 0$ so we get saddle point. At $(2,1)$, $D(2,1) > 0$, and $f_{xx}(2,1) > 0$, so we get local min. □