MATH 32A DISCUSSION

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1. Limits and Continuity

1.1. **Definitions.** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then the *limit* $\lim_{(x,y)\to(a,b)} f(x, y) = L$ exists if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$, and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$.

It is *continuous* at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

1.2. **Exercise 15.2.5.** Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2).$$

Solution. Recall that polynomials are continuous, hence we may employ the substitution method to get the limit: $5 \cdot 1^3 - 1^2 \cdot 2^2 = 1$. □

1.3. **Exercise 15.2.16.** Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}.$$

Solution. Notice that $0 \le \frac{x^2}{x^2+2y^2} \le 1$, so the quantity above is squeezed between 0 and $\sin^2 y \to 0$, hence the limit is 0. □

1.4. **Exercise 15.2.38.** Determine the set of points at which the function is continuous:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+xy+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Notice that $x^2 + xy + y^2 = 0$ only at $(x, y) = (0, 0)$. Indeed, the minimum with respect to x is when $x = -y/2$, which makes the expression $3y^2/4$. Therefore the function is continuous at all points besides the origin.

As for the origin, it is not continuous there. Indeed, along the line $y = x$ the limit is $\lim_{x\to0} \frac{x^2}{x^2+x^2+x^2} = \frac{1}{3}$. Whereas along the line $y = 2x$ the limit is $\lim_{x\to0} \frac{2x^2}{x^2+2x^2+4x^2} = \frac{2}{7}$. □

1.5. **Exercise 15.2.43.** Discuss the continuity of the function

$$f(x, y) = \begin{cases} \frac{\sin xy}{xy} & \text{if } xy \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Solution. Obviously f is continuous at all $xy \neq 0$. Now fix a point (x_0, y_0) such that $x_0 y_0 = 0$. Take an arbitrary continuous path $C(t) = (x(t), y(t))$ such that $C(0) = (x(0), y(0)) = (x_0, y_0)$, and let $z(t) = x(t)y(t)$. If there is a small neighbourhood around 0 such that the path lies in the set $xy = 0$, then the limit is 1. Otherwise, we may assume $f(C(t)) = \sin(x(t)y(t))/(x(t)y(t)) = \sin(z(t))/z(t)$. Then we seek the limit $\lim_{t\to 0} f(x(t), y(t)) = \lim_{t\to 0} \sin(z(t))/z(t))$. But as $t \to 0$, we have $z(t) \to z(0) = x(0)y(0) = 0$, so the limit is 1 as $\lim_{z\to 0} \sin z/z = 1$. Hence we conclude that f is continuous everywhere.

Alternatively, consider $g(x, y) = xy$ and $h(t) = \sin t/t$ for $t \neq 0$ and $h(t) = 1$ for $t = 0$. Then $f(x, y) = h(g(x, y))$, and both h and g are continuous everywhere, hence so is f. □

1.6. **Exercise 15.2.45.** Show that the function f given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

Solution. First recall that $\mathbf{a} \cdot \mathbf{x} = |\mathbf{a}|\,|\mathbf{x}| \cos\theta$, where θ is the angle between them. Therefore $\mathbf{a}\cdot\mathbf{x} \leq |\mathbf{a}|\,|\mathbf{x}|$. Now compute $|f(\mathbf{x}) - f(\mathbf{a})| = ||\mathbf{x}| - |\mathbf{a}|| = \sqrt{(|\mathbf{x}| - |\mathbf{a}|)^2} = \sqrt{|\mathbf{x}|^2 - 2\,|\mathbf{x}|\,|\mathbf{a}| + |\mathbf{a}|^2} \leq \sqrt{\mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{a}} = \sqrt{(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})} = |\mathbf{x}-\mathbf{a}|$. So if we make $\mathbf{a}$ approach $\mathbf{x}$, namely, $|\mathbf{x}-\mathbf{a}| \to 0$, then $|f(\mathbf{x}) - f(\mathbf{a})| \to 0$, namely, $f(\mathbf{x})$ approaches $f(\mathbf{a})$, as desired. □

2. Partial Derivatives

2.1. **Definitions.** The partial derivative of f with respect to x at (a, b) is $f_x(a, b) = g'(a)$ where $g(x) = f(x, b)$.

2.2. **Exercise 15.3.16.** Find the first partial derivatives of $f(x, y) = x^4y^3 + 8x^2y$.

Solution. Treating y constant we get $f_x(x, y) = 4x^3y^3 + 16xy$. Treating x constant we get $f_y(x, y) = 3x^4y^2 + 8x^2$. □

2.3. **Exercise 15.3.28.** Find the first partial derivatives of

$$f(x, y) = \int_y^x \cos(t^2)\, dt.$$

Solution. Let $A(x) = \int_0^x \cos(t^2)dt$, by FTC, we get $A'(x) = \cos(x^2)$. So $f(x, y) = A(x) - A(y)$, giving $f_x(x, y) = A'(x) = \cos(x^2)$ and $f_y(x, y) = -A'(y) = -\cos(y^2)$. □

2.4. **Exercise 15.3.45.** Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given

$$x^2 + y^2 + z^2 = 3xyz.$$

Solution. Applying $\frac{\partial}{\partial x}$ we get $2x + 2z\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x}$, solving, we get $\frac{\partial z}{\partial x} = \frac{3yz-2x}{2z-3xy}$. Similarly, $\frac{\partial z}{\partial y} = \frac{3xz-2y}{2z-3xy}$. □

2.5. **Exercise 15.3.94.** If $f(x, y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0, 0)$.

Solution. Let $g(x) = f(x, 0) = x$, so $g'(x) = 1$. Now $f_x(0, 0) = g'(0) = 1$. Notice that it would be much more difficult to calculate $f_x(x, y)$ directly with y an arbitrary constant. □