MATH 32A DISCUSSION

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1. Motion in Space

1.1. **Exercise 14.4.9.** Find the velocity, acceleration, and speed of a particle with position function $\mathbf{r}(t) = \langle t^2+1, t^3, t^2-1 \rangle$.

Solution. We have $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2, 2t \rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 2 \rangle$, and $v(t) = |\mathbf{v}(t)| = \sqrt{4t^2+9t^4+4t^2}$. □

1.2. **Exercise 14.4.17.** Find the position vector of a particle that has the given accleration and the specified initial velocity and position. $\mathbf{a}(t) = \langle 2t, \sin t, \cos 2t \rangle$, $\mathbf{v}(0) = \mathbf{i}$, $\mathbf{r}(0) = \mathbf{j}$.

Solution. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, $\mathbf{v}(t) = \int \mathbf{a}(t)dt = \langle t^2 + c_1, -\cos t + c_2, \frac{1}{2}\sin 2t + c_3 \rangle$, with $\mathbf{v}(0) = \langle c_1, -1+c_2, c_3 \rangle = \langle 1, 0, 0 \rangle$, so $c_1 = c_2 = 1$, $c_3 = 0$, thus $\mathbf{v}(t) = \langle t^2+1, 1-\cos t, \frac{1}{2}\sin 2t \rangle$.

Repeating, $\mathbf{r}(t) = \int \mathbf{v}(t)dt = \langle \frac{1}{3}t^3 + t + c_1, t - \sin t + c_2, -\frac{1}{4}\cos 2t + c_3 \rangle$, with $\mathbf{r}(0) = \langle c_1, c_2, -\frac{1}{4} + c_3 \rangle = \langle 0, 1, 0 \rangle$, so $c_1 = 0$, $c_2 = 1$, and $c_3 = \frac{1}{4}$, yielding $\mathbf{r}(t) = \langle \frac{1}{3}t^3 + t, t - \sin t + 1, \frac{1}{4} - \frac{1}{4}\cos 2t \rangle$. □

1.3. **Exercise 14.4.20.** What force is required so that a particle of mass m hass the position function $\mathbf{r}(t) = \langle t^3, t^2, t^3 \rangle$.

Solution. We have $\mathbf{F} = m\mathbf{a}$, so $\mathbf{r}'(t) = \langle 3t^2, 2t, 3t^2 \rangle$, $\mathbf{r}''(t) = \langle 6t, 2, 6t \rangle$, and $\mathbf{F}(t) = \langle 6mt, 2m, 6mt \rangle$. □

1.4. **Exercise 14.4.22.** Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

Solution. Constant speed means $v'(t) = 0$. Now $(v(t))^2 = |\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$. Differentiate both sides: $2v(t)v'(t) = \mathbf{v}(t) \cdot \mathbf{a}(t) + \mathbf{a}(t) \cdot \mathbf{v}(t)$. Thus we conclude that $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$. □

1.5. **Exercise 14.4.37.** Find the tangential and normal components of the acceleration vector of $\mathbf{r}(t) = \langle e^t, \sqrt{2}t, e^{-t} \rangle$.

Solution. The tangential component is $a_T = v'$ and the normal component is $a_N = \kappa v^2$. Now $\mathbf{v}(t) = \langle e^t, \sqrt{2}, -e^{-t} \rangle$, so $v(t) = |\mathbf{v}(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$, hence $a_T = v' = e^t - e^{-t}$. Also, $\mathbf{T}(t) = \mathbf{v}(t)/v(t) = \langle e^t, \sqrt{2}, -e^{-t} \rangle / (e^t + e^{-t})$. So we get $\mathbf{T}'(t) = (\mathbf{a}(t)v(t) - \mathbf{v}(t)v'(t))/(v(t))^2 = ((e^t + e^{-t})\langle e^t, 0, e^{-t} \rangle - (e^t - e^{-t})\langle e^t, \sqrt{2}, -e^{-t} \rangle)/(e^t + e^{-t})^2$ Now $|\mathbf{T}'(t)| = \kappa v$, so $a_N = |\mathbf{T}'(t)|\, v(t)$. □

2. Functions of Several Variables

2.1. **Exercise 15.1.9.** Let $f(x,y,z) = e^{\sqrt{z-x^2-y^2}}$. Find the domain and range of f.

Solution. We need $z - x^2 - y^2 \geq 0$, so the domain is a solid paraboloid. The range of $\sqrt{z - x^2 - y^2}$ is $[0, \infty)$ so the range of f is $[1, \infty)$. □

2.2. **Exercise 15.1.15.** Find and sketch the domain of the function $f(x,y) = \sqrt{1-x^2} - \sqrt{1-y^2}$.

Solution. We need $1 - x^2 \geq 0$ so $-1 \leq x \leq 1$, similarly, $-1 \leq y \leq 1$, so the domain is a square. □

2.3. **Exercise 15.1.65–66.** Describe how the graph of g is obtained from the graph of f.

(a) $g(x,y) = f(x,y) + 2$,
(b) $g(x,y) = 2f(x,y)$,
(c) $g(x,y) = -f(x,y)$,
(d) $g(x,y) = 2 - f(x,y)$;
(e) $g(x,y) = f(x-2,y)$,
(f) $g(x,y) = f(x,y+2)$,
(g) $g(x,y) = f(x+3,y-4)$.

Solution. This is obvious. □

3. Cylinders and Quadric Surfaces

3.1. **Exercise 13.6.3.** Describe and sketch the surface $y^2 + 4z^2 = 4$.

Solution. Elliptic cylinder. □

3.2. **Exercise 13.6.42.** Sketch the region bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

Solution. Solve for intersection: $x^2 + y^2 = 2 - x^2 - y^2$, $x^2 + y^2 = 1$ is a circle, at $z = 1$. □

3.3. **Exercise 13.6.43.** Find an equation of the surface obtained by rotating the parabola $y = x^2$ about the y-axis.

Solution. We get $y = x^2 + z^2$. □