MATH 31A DISCUSSION

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1. Basic Integration

1.1. **Exercise 5.1.71.** Describe the area represented by the limit

$$\lim_{N\to\infty} \frac{3}{N} \sum_{j=1}^{N} \left(2 + \frac{3j}{N}\right)^4 .$$

Solution. Consider the area under the curve $f(x)$ from $x = a$ to $x = b$. We divide the interval $b - a$ into N equal slots, so each slot has width $\Delta x = \frac{b-a}{N}$. The right end points of the jth rectangle is $a + j\Delta x$. The value of the function at the right end point of the jth rectangle is therefore $f(a+j\Delta x)$. If we use these as the height of the rectangles, then the area of the jth rectangle is $\Delta x \cdot f(a + j\Delta x)$. We sum this up to get $\sum_{j=1}^{N} \Delta x \cdot f(a+j\Delta x)$. We can factor out the width of the rectangles as they are all the same. Thus we get $\Delta x \sum_{j=1}^{N} f(a + j\Delta x)$. If we let N goes to infinity, the approximation becomes finer and finer, and approaches the area under the curve.

Here $a = 2$, $b-a = 3$ so $b = 5$, and $f(x) = x^4$. Thus the limit represents the area between the graph of x^4 and the x-axis, from the interval $[2, 5]$. In other words, $\int_2^5 x^4\, dx$. □

1.2. **Exercise 5.1.73.** Evalute $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{j}{N}\right)^2}$ by interpreting it as the area of part of a familair geometric figure.

Solution. Guess $f(x) = \sqrt{1 - x^2}$, $a = 0$, $b - a = 1$, so $b = 1$. Thus the limit represents the area between the graph of $\sqrt{1 - x^2}$ and the x-axis, from the interval $[0, 1]$. In other words, $\int_0^1 \sqrt{1 - x^2}\, dx$. This is a fourth of a unit circle, hence the limit is $\pi/4$. □

1.3. **Exercise 5.2.80.** Verify by interpreting the integral as an area:

$$\int_0^b \sqrt{1 - x^2}\, dx = \frac{1}{2} b\sqrt{1 - b^2} + \frac{1}{2}\theta,$$

where $0 \le b \le 1$ and θ is the angle between 0 and $\frac{\pi}{2}$ such that $\sin\theta = b$.

Solution. The integral represents the area under the top half of the unit circle from 0 to b. We separate the area into two parts: a triangle with base b and height $\sqrt{1 - b^2}$, yielding the term $\frac{1}{2}b\sqrt{1 - b^2}$, and the circular sector of radius 1 and central angle θ, yielding the term $\frac{1}{2}\theta = \pi r^2 \cdot \frac{\theta}{2\pi}$. □

2. Bonus

2.1. **Minimal Distance.** Let A, B, and C be three points on a plane. Suppose C lies on the perpendicular bisector of $\overline{AB}$. Find a point P which minimises the sum of the distances $\overline{AP} + \overline{BP} + \overline{CP}$.

Solution. Let $2L$ be the length of $\overline{AB}$ and c be the distance from C to $\overline{AB}$. We first prove that point P also lies on the perpendicular bisector. Indeed, let Q be the closest point to P on the perpendicular bisector. Let x be the distance from $\overline{AB}$ to Q, and y the distance from P to Q. The total length $z = \sqrt{(L-y)^2 + x^2} + \sqrt{(L+y)^2 + x^2} + \sqrt{(c-x)^2 + y^2}$. Notice that whatever x is, $y = 0$ minimises z. Indeed, $\sqrt{(c-x)^2 + y^2}$ is obviously minimised when $y = 0$. As for the sum of the first two, taking the derivative with respect to y gives $\frac{-y}{\sqrt{(L-y)^2+x^2}} + \frac{y}{\sqrt{(L+y)^2+x^2}}$. Obviously $y = 0$ is a critical point. If $y \neq 0$, then $(L-y)^2 = (L+y)^2$, yielding the same result. So we conclude that $y = 0$, hence $P = Q$ is on the perpendicular bisector, as claimed.

Now then z simplifies to $z = 2\sqrt{L^2 + x^2} + c - x$. Minimising with respect to x gives $\frac{dz}{dx} = \frac{2x}{\sqrt{L^2+x^2}} - 1$. So the critical point is $x = L/\sqrt{3}$. It remains to check the end points $x = 0$ and $x = c$, and compare these values depending on the ratio of c/L. This is left as an exercise to the reader. □

2.2. **Piano Competition.** 1000 people from all over the country are to gather and have a piano competition. 600 of the participants live in Seattle. They claim that to minimise total travel distance, the competition should be held in Seattle itself. The other 400 participants are furious and think they should calculate the average longitude and lattitude instead. Which side is correct?