MATH 31A DISCUSSION

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Applications of the Derivative

1. Related Rates

1.1. **Exercise 3.9.44.** A wheel of radius r is centred at the origin. As it rotates, the rod of length L attached at the point P drives a piston back and forth in a straight line. Let x be the distance from the origin to the point Q at the end of the rod.

(a) Use the Pythagorean Theorem to show that
$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$
(b) Differentiate part (a) with resepct to t to prove that
$$2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0.$$
(c) Calculate the speed of the piston when $\theta = \frac{\pi}{2}$, assuming that $r = 10$ cm, $L = 30$ cm, and the wheel rotates at 4 re volutions per minute.

Solution. Parts (a) and (b) are straightforward. 4 revolutions per minute means $\frac{d\theta}{dt} = 4\cdot 2\pi$ per minute. From part (a), we get $30^2 = x^2 + 10^2$, so $x = 20\sqrt{2}$. Plugging in, we get $2(20\sqrt{2} - 0)(\frac{dx}{dt} + 10\cdot 8\pi) + 0 = 0$. So $\frac{dx}{dt} = -80\pi$ cm per minute. □

2. Linear Approximations

2.1. **Approximating Change.** If f is differentiable at $x = a$ and Δx is small, then
$$\Delta f \approx f'(a)\Delta x$$
where $\Delta f = f(a + \Delta x) - f(a)$.

2.2. **Linearisation.** If f is differentiable at $x = a$, and x is close to a, then
$$f(x) \approx L(x) = f'(a)(x - a) + f(a).$$

3. Extrema

3.1. **Basics.**

3.1.1. *Critical Points.* A number c in the domain of f is called a *critical point* if either $f'(c) = 0$ or $f'(c)$ does not exist.

3.1.2. *Local Extrema.* If $f(c)$ is a local extremum, then c is a critical point of f.

3.1.3. *Extrema on Closed Interval.* If $f(x)$ is continuous on $[a, b]$, and $f(c)$ be an extremum on $[a, b]$. Then c is either a critical point or one of the endpoints a or b.

3.1.4. *Rolle's Theorem.* Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c between a and b such that $f'(c) = 0$.

3.2. **Exercise 4.2.39.** Find the maximum and minimum values of $y = \sin x \cos x$ on $[0, \frac{\pi}{2}]$.

Solution. Notice $y' = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$. If $y' = 0$, then $\cos^2 x = \frac{1}{2}$, so $\cos x = \pm\frac{\sqrt{2}}{2}$. In $[0, \frac{\pi}{2}]$, this occur at $x = \frac{\pi}{4}$. Now $f(0) = f(\frac{\pi}{2}) = 0$, and $f(\frac{\pi}{4}) = \frac{1}{2}$. So min is 0 and max is $\frac{1}{2}$. □

3.3. **Exercise 4.2.73–74.** Show that $f(x) = x^2 - 2x + 3$ takes on only positive values. Find conditions on r and s under which the quadratic function $f(x) = x^2 + rx + s$ takes on only positive values. Show that if f takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

Solution. For $f(x) = x^2 - 2x + 3$, we have $f'(x) = 2x - 2$, so $x = 1$ is critical point, and $f(1) = 2 > 0$. More generally, for $f(x) = x^2 + rx + s$, we have $f'(x) = 2x + r$, so $x = -\frac{r}{2}$ is critical point. Now $f(-\frac{r}{2}) = s - \frac{r^2}{4}$. So if we want this to be positive, we must have $s > \frac{r^2}{4}$. If f takes on both positive and negative values, then the roots are $x = \frac{-r \pm \sqrt{r^2 - 4s}}{2}$, whose midpoint is $x = -\frac{r}{2}$, as desired. □