MATH 31A DISCUSSION

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1. Derivatives

1.1. **Basics.** Given a function $f(x)$. The slope of the tangent line at $x = c$ is $f'(c)$.

1.1.1. *Power Rule.* For all exponents $n \in \mathbb{R}$, $\frac{d}{dx}x^n = nx^{n-1}$. Not for e^x, x^x.

1.1.2. *Linearity Rules.* If f and g are differentiable functions, $c \in \mathbb{R}$, then cf and $f + g$ are differentiable. Indeed, $(f + g)' = f' + g'$ and $(cf)' = cf'$.

1.1.3. *Product and Quotient Rules.* If f and g are differentiable, $(fg)' = fg' + gf'$. And $(f/g)' = (f'g - g'f)/g^2$.

1.2. **Exercise 3.2.46.** Sketch the graphs of $f(x) = x^2 - 5x + 4$ and $g(x) = -2x + 3$. Find the value of x at which the graphs have parallel tangent lines.

Solution. We need $f'(x) = g'(x)$. Notice $f'(x) = 2x - 5$ and $g'(x) = -2$. So we solve $2x - 5 = -2$ to get $x = \frac{3}{2}$. □

1.3. **Exercise 3.2.52.** Show that if the tangent lines to the graph of $y = \frac{1}{3}x^3 - x^2$ at $x = a$ and $x = b$ are parallel, then either $a = b$ or $a + b = 2$.

Solution. We want $y'(a) = y'(b)$. Now $y' = x^2 - 2x$. So if $a^2 - 2a = b^2 - 2b$, then $(a^2 - b^2) = 2(a - b)$, giving $(a + b)(a - b) = 2(a - b)$. □

1.4. **Exercise 3.3.55.** Let $f(x)$ be a polynomial. Then c is a root of $f(x)$ if and only if $f(x) = (x - c)g(x)$ for some polynomial $g(x)$. We say that c is a multiple root if $f(x) = (x - c)^2h(x)$ for some polynomial $h(x)$.

Show that c is a multiple root of $f(x)$ if and only if c is a root of both $f(x)$ and $f'(x)$.

Solution. Suppose c is a multiple root of $f(x)$. Then there exists some polynomial $h(x)$ such that $f(x) = (x - c)^2h(x)$. Obviously c is a root of $f(x)$. Let $g(x) = (x - c)h(x)$, thus $f(x) = (x - c)g(x)$. Using the Product Rule, we have $f'(x) = g(x) + (x - c)g'(x)$. Notice $f'(c) = g(c) = 0$, so c is a root of $f'(x)$, as desired.

Conversely, suppose c is a root of both $f(x)$ and $f'(x)$. As c is a root of $f(x)$, there exists some polynomial $g(x)$ such that $f(x) = (x - c)g(x)$. By the Product Rule, we again have $f'(x) = g(x) + (x - c)g'(x)$ and $f'(c) = g(c)$. Since c is a root of $f'(x)$, we have $f'(c) = 0$, hence $g(c) = 0$. We conclude that $g(x) = (x - c)h(x)$ for some polynomial $h(x)$, hence $f(x) = (x - c)^2h(x)$, as desired. □

1.5. **Exercise 3.3.56.** Use Exercise 55 to determine whether $c = -1$ is a multiple root of the polynomial $f(x) = x^4 + x^3 - 5x^2 - 3x + 2$.

Solution. First check $f(-1) = 0$, so -1 is a root of f. Now $f'(x) = 4x^3 + 3x^2 - 10x - 3$. So $f'(-1) = 6 \neq 0$, so -1 is *not* a multiple root. □

1.6. **Exercise 3.4.32.** It takes a stone 3 s to hit the ground when dropped from the top of a building. How high is the building and what is the stone's velocity upon impact.

Solution. The position is $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$, where s_0 is the initial height, v_0 is the initial velocity, and $g \approx 9.8$ m/s^2. We have $v_0 = 0$, and $s(3) = 0$. So we get $s(3) = s_0 - \frac{1}{2}g(3)^2 = 0$. So the initial height is $4.5g \approx 44.1$ m.

The velocity is $v(t) = v_0 - gt$. So $v(3) = -3g \approx -29.4$ m/s. □

1.7. **Exercise 3.4.33.** A ball is tossed up vertically from ground level and returns to earth 4 s later. What was the initial velocity of the stone and how high did it go?

Solution. We have $s_0 = 0$, $s(4) = 0$, so we can solve for v_0. Indeed, $s(4) = 0 + 4v_0 - 8g = 0$, so $v_0 = 2g = 19.6$ m/s.

The maximum height occurs when the derivative is zero. So $s'(t) = v(t) = v_0 - gt = 0$ gives $t = v_0/g = 2$. This confirms what we thought this is the half way point. The height is $s(2) = 0 + 2v_0 - 2g = 2g = 19.6$ m. □