MATH 31A DISCUSSION

JED YANG

1. Introduction

Lecture 1

- Instructor: Steve Butler.
- Location: HAINES 39.

Sections 1A and 1B

- Email: `mailto:jedyang@ucla.edu`.
- Office: MS 6617A.
- Office hours: R 12:30–13:30.
- Discussion Location: MS 5117 (T) and 5138 (R).
- Website: `http://www.math.ucla.edu/~jedyang/31a.1.10w/`.
- SMC: Jan. 6–Mar. 11, M–R 09:00–15:00, MS 3974, T 12:00–13:00.

2. Administration

- HW due Fridays in lecture, can turn in early to me, and I will hand back in section.
- Textbook: Rogawski, Single Variable Calculus, 2008.
- Confirm office hour.

3. Precalculus Review

3.1. **Exercise 1.2.21.** Find the equation of the perpendicular bisector of the segment joining $(1, 2)$ and $(5, 4)$.

Solution. Slope of segment is $m_1 = \frac{4-2}{5-1} = \frac{1}{2}$. Slope of perpendicular bisector is $m_2 = -1/m_1 = -2$. Mid point is $(\frac{1+5}{2}, \frac{2+4}{2})$. So the equation can be written as $y - 3 = -2(x - 3)$. □

3.2. **Exercise 1.2.23.** Find the equation of the line with x-intercept $x = 4$ and y-intercept $y = 3$.

Solution. Equation of the line is $y = mx+b$, where b is the y-intercept, hence $b = 3$. The x-intercept $x = 4$ will yield $y = 0$ (by definition), so substituting, we may solve for m. We get $0 = 4m + 3$, hence $m = -\frac{3}{4}$. So the equation can be written as $y = -\frac{3}{4}x + 3$. □

3.3. **Exercise 1.2.24.** A line of slope $m = 2$ passes through $(1, 4)$. Find y such that $(3, y)$ lies on the line.

Solution. One way is to write down an equation of the line in point-slope form: $y = 2(x-1)+4$. Then we see clearly that if $x = 3$, then $y = 8$. Alternatively, the slope m is the change of y over the change of x. Symbolically, $m = \frac{\Delta y}{\Delta x}$, or $\Delta y = m\Delta x$. This concept will be useful later when we deal with differentials $dy = m\,dx$. Since the change in x is $\Delta x = 3-1 = 2$, we get that the change in y is $\Delta y = y - 4 = 2\cdot 2 = 4$, hence $y = 8$. This method seems longer, but conceptually it is easier to do in one's head, and will lead to intution for calculus later. □

3.4. **Exercise 1.4.55.** Use the addition formulae for sine and cosine to prove

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \tag{1}$$

$$\cot(a-b) = \frac{\cot a \cot b + 1}{\cot b - \cot a}. \tag{2}$$

Proof. Recall that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{3}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b. \tag{4}$$

Now

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} \tag{5}$$

$$= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} \tag{6}$$

$$= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a}{\cos a}\frac{\sin b}{\cos b}} \tag{7}$$

$$= \frac{\tan a + \tan b}{1 - \tan a \tan b}, \tag{8}$$

where we get from (6) to (7) by dividing top and bottom by $\cos a \cos b$.

The case for cotangent is completely analogous. Remember $\cot x = \frac{\cos x}{\sin x}$ and that $\sin(-b) = -\sin(b)$ and $\cos(-b) = \cos(b)$. Work out the details and convince yourself. □

3.5. **Exercise 1.4.56.** Let θ be the angle between the line $y = mx + b$ and the x-axis. Prove that $m = \tan\theta$.

Proof. This is trivial. □

3.6. **Exercise 1.4.57.** Let L_1 and L_2 be the lines of slope m_1 and m_2, respectively. Show that the angle θ between L_1 and L_2 satisfies $\cot\theta = \frac{m_2 m_1 + 1}{m_2 - m_1}$.

Proof. This is immediate by using Exercises 55 and 56. □

3.7. **Exercise 1.4.58. Perpendicular Lines.** Use Exercise 57 to prove that two lines with nonzero slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

Proof. What is $\cot(\pi/2)$? □

3.8. **Exercise 1.4.59.** Apply the double-angle formula to prove:

(a) $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$.

(b) $\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$.

Guess the values of $\cos\frac{\pi}{32}$ and of $\cos\frac{\pi}{2^n}$ for all n.

Proof. Recall $\cos^2 t = \frac{1+\cos(2t)}{2}$. For the general case, let $a_0 = 0$ and define inductively $a_n = \sqrt{2+a_{n-1}}$. We claim that for $n \geq 1$, we have $\cos\frac{\pi}{2^n} = \frac{1}{2}a_{n-1}$. The base case is trivial. By induction, assume $\cos\frac{\pi}{2^n} = \frac{1}{2}a_{n-1}$. By the half-angle formula, we get $\cos\frac{\pi}{2^{n+1}} = \sqrt{\frac{1}{2}(1+\frac{1}{2}a_{n-1})} = \sqrt{\frac{1}{4}(2+a_{n-1})} = \frac{1}{2}a_n$. □