MATH 31A DISCUSSION

JED YANG

1. Theorems

1.1. **Fundamental Theorem of Calculus, II.** Assume that $f(x)$ is continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,dt.$$

Then A is an intiderivative of f, that is, $A'(x) = f(x)$, or equivalently

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

1.2. **The Substitution Method.** If $F'(x) = f(x)$, then

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C.$$

2. More Topics in Integration

2.1. **Exercise 5.4.40.** Find the smallest positive critical point of

$$F(x) = \int_0^x \cos(t^{3/2})\,dt$$

and determine whether it is a local min or max.

Solution. As usual, to find critical point, we take derivative and set to 0. By FTC2, we get $F'(x) = \cos(x^{3/2})$. The smallest positive zero is when $x^{3/2} = \frac{\pi}{2}$. So $x = (\pi/2)^{2/3}$ is the smallest positive critical point. Since F' goes from positive to negative, it is a local max. □

2.2. **Exercise 5.6.50.** Evaluate the indefinite integral

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx.$$

Solution. Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\,dx$. So $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$. □

2.3. **Exercise 5.6.54. Can They Both Be Right?** Use $u = \tan x$ and $u = \sec x$ to evaluate

$$\int \tan x \sec^2 x\,dx.$$

Show that these yield different answers and explain the apparent contradiction.

Solution. If $u = \tan x$ then $du = \sec^2 x\,dx$, so the integral becomes $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$. On the other hand, if $u = \sec x$ then $du = \tan x \sec x\,dx$, so the integral becomes $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sec^2 x + C$. Notice however that $\sec^2 x = \tan^2 x + 1$. $\square$

2.4. **Exercise 5.6.72.** Evaluate

$$\int_0^2 r\sqrt{5 - \sqrt{4 - r^2}}\,dr.$$

Solution. Let $u = 5 - \sqrt{4 - r^2}$, then $du = \frac{r}{\sqrt{4-r^2}}\,dr = \frac{r}{5-u}\,dr$. So the integral becomes $\int_3^5 (5-u)\sqrt{u}\,du = \int_3^5 5u^{1/2} - u^{3/2}\,du = \frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\Big|_3^5 = \frac{20\sqrt{5}}{3} - \frac{32\sqrt{3}}{5}$. $\square$