# MATH 31A DISCUSSION

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## 1. Theorems

**1.1. Comparison Theorem.** If $g(x) \leq f(x)$ on an interval $[a, b]$, then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx.$$

**1.2. Fundamental Theorem of Calculus, I.** Assume that $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

## 2. Topics in Integration

**2.1. Exercise 5.2.80.** Verify by interpreting the integral as an area:

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2}b\sqrt{1-b^2} + \frac{1}{2}\theta,$$

where $0 \leq b \leq 1$ and $\theta$ is the angle between 0 and $\frac{\pi}{2}$ such that $\sin\theta = b$.

*Solution.* The integral represents the area under the top half of the unit circle from 0 to $b$. We separate the area into two parts: a triangle with base $b$ and height $\sqrt{1-b^2}$, yieling the term $\frac{1}{2}b\sqrt{1-b^2}$, and the circular sector of radius 1 and central angle $\theta$, yielding the term $\frac{1}{2}\theta = \pi r^2 \cdot \frac{\theta}{2\pi}$. □

**2.2. Exercise 5.3.39.** Write the integral $\int_0^\pi |\cos x|\ dx$ as a sum of integrals without absolute values and evaluate.

*Solution.* Notice that $\cos x$ is nonnegative on $[0, \pi/2]$ and nonpositive on $[\pi/2, \pi]$. As such, the integral in question is $\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi/4} -\cos x\,dx = \sin x \mid_0^{\pi/2} - \sin x \mid_{\pi/2}^{\pi} = 1 - 0 - 0 + 1 = 2$. □

**2.3. Exercise 5.3.52.** Apply the Comparison Theorem to the inequality $\sin x \leq x$ (valid for $x \geq 0$) to prove $1 - \frac{x^2}{2} \leq \cos x \leq 1$. Apply it again to prove $x - \frac{x^3}{6} \leq \sin x \leq x$ (for $x \geq 0$).

*Solution.* On $[0, t]$ for some $t > 0$, we have $\sin x \leq x$. By the Comparison Theorem, we get $\int_0^t \sin x\,dx \leq \int_0^t x\,dx$. This gives $-\cos t + 1 \leq t^2/2$ hence $1 - \frac{t^2}{2} \leq \cos t$. Since cos is even, $\cos -t = \cos t$ satisfies the same inequality. Applying this again we get $\int_0^t 1 - \frac{x^2}{2}\,dx \leq \int_0^t \cos x\,dx$, yielding $t - \frac{t^3}{3} \leq \sin t$, as desired. □