MATH 31A DISCUSSION

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1. Optimisation

1.1. **Exercise 4.6.15.** Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius r.

Solution. Let x and y be the side lengths. By Pythagorean theorem, they satisfy $x^2 + y^2 = 4r^2$. We want to maximize $A = xy$. Solve for $y = \sqrt{4r^2 - x^2}$. So $A = x\sqrt{4r^2 - x^2}$. We can maximize $A^2 = x^2(4r^2 - x^2) = 4r^2x^2 - x^4$. Differentiate to get $(A^2)' = 8r^2x - 4x^3$. Critical points are $8r^2x - 4x^3 = 0$ so $x = 0$ or $x^2 = 2r^2$, that is, $x = r\sqrt{2}$. End points $x = 0, 2r$. We get $A(0) = A(2r) = 0$ and $A(r\sqrt{2}) = 2r^2$. So area is maximized when $x = y = r\sqrt{2}$, that is, we have a square. □

1.2. **Catching a bus.** I am at one corner of a rectangular park with sides 60 and 300 metres. The bus stop is at the opposite corner. I can run on the grass at 5 m/s, and I can skateboard on the sidewalk (along the long side) at 13 m/s. I want to get to the bus stop by running across the park and then skateboarding on the sidewalk. Where should I run to in order to get there the fastest?

Solution. Let x be the distance that I skip on the long sidewalk. In otherwords, I run $\sqrt{60^2 + x^2}$ on grass, and then run on sidewalk for $300 - x$. Dividing by the rates, we get the total travel time is $y = \frac{300-x}{13} + \frac{\sqrt{60^2+x^2}}{5}$. Differentiating, we get $y' = -\frac{1}{13} + \frac{x}{5\sqrt{60^2+x^2}}$. Setting equal to 0, we solve and get $x = 25$.

Note that if there were a sidewalk on the short side too, then it would be faster to skate along the sidewalk for the entire time instead. □

2. Basic Integration

2.1. **Exercise 4.8.58.** Let $f''(t) = t - \cos t$, $f'(0) = 2$, $f(0) = -2$. Find f' and f.

Solution. By integrating, we see $f'(t) = t^2/2 - \sin t + c$. Since $f'(0) = c = 2$, we get $f'(t) = t^2/2 - \sin t + 2$. Integrating again to get $f(t) = t^3/6 + \cos t + 2t + d$. Since $f(0) = 1 + d = -2$, we get $d = -3$, yielding $f(t) = t^3/y + \cos t + 2t - 3$. □

2.2. **Exercise 4.8.65.** A car traveling 84 ft/s begins to decelerate at a constant rate of 14 ft/s^2. After how many seconds does the car come to a stop and how far will the car have travelled before stopping?

Solution. Let $v(t)$ be the velocity and $a(t)$ be the acceleration. Then $a(t) = v'(t) = -14$ and $v(0) = 84$. We seek t such that $v(t) = 0$. Integrating $a(t)$, we get $v(t) = -14t + c$. Since $v(0) = 84$, we get $c = 84$ and $v(t) = 84 - 14t$. So if $v(t) = 0$ then $t = 6$.

Next, integrating $v(t)$ to get $s(t) = 84t - 7t^2 + c$, the position. The distance travelled is therefore $s(6) - s(0) = 252$. Note c here is irrelevant. □

2.3. **Exercise 5.1.71.** Describe the area represented by the limit

$$\lim_{N\to\infty} \frac{3}{N} \sum_{j=1}^{N} \left(2 + \frac{3j}{N}\right)^4 .$$

Solution. Consider the area under the curve $f(x)$ from $x = a$ to $x = b$. We divide the interval $b - a$ into N equal slots, so each slot has width $\Delta x = \frac{b-a}{N}$. The right end points of the jth rectangle is $a + j\Delta x$. The value of the function at the right end point of the jth rectangle is therefore $f(a+j\Delta x)$. If we use these as the height of the rectangles, then the area of the jth rectangle is $\Delta x \cdot f(a + j\Delta x)$. We sum this up to get $\sum_{j=1}^{N} \Delta x \cdot f(a+j\Delta x)$. We can factor out the width of the rectangles as they are all the same. Thus we get $\Delta x \sum_{j=1}^{N} f(a + j\Delta x)$. If we let N goes to infinity, the approximation becomes finer and finer, and approaches the area under the curve.

Here $a = 2$, $b-a = 3$ so $b = 5$, and $f(x) = x^4$. Thus the limit represents the area between the graph of x^4 and the x-axis, from the interval $[2, 5]$. In other words, $\int_2^5 x^4 dx$. □

3. Bonus

3.1. **Minimal Distance.** Let A, B, and C be three points on a plane. Suppose C lies on the perpendicular bisector of $\overline{AB}$. Find a point P which minimises the sum of the distances $\overline{AP} + \overline{BP} + \overline{CP}$.

Solution. Let $2L$ be the length of $\overline{AB}$ and c be the distance from C to $\overline{AB}$. We first prove that point P also lies on the perpendicular bisector. Indeed, let Q be the closest point to P on the perpendicular bisector. Let x be the distance from $\overline{AB}$ to Q, and y the distance from P to Q. The total length $z = \sqrt{(L-y)^2 + x^2} + \sqrt{(L+y)^2 + x^2} + \sqrt{(c-x)^2 + y^2}$. Notice that whatever x is, $y = 0$ minimises z. Indeed, $\sqrt{(c-x)^2 + y^2}$ is obviously minimised when $y = 0$. As for the sum of the first two, taking the derivative with respect to y gives $\frac{-y}{\sqrt{(L-y)^2+x^2}} + \frac{y}{\sqrt{(L+y)^2+x^2}}$. Obviously $y = 0$ is a critical point. If $y \neq 0$, then $(L-y)^2 = (L+y)^2$, yielding the same result. So we conclude that $y = 0$, hence $P = Q$ is on the perpendicular bisector, as claimed.

Now then z simplifies to $z = 2\sqrt{L^2 + x^2} + c - x$. Minimising with respect to x gives $\frac{dz}{dx} = \frac{2x}{\sqrt{L^2+x^2}} - 1$. So the critical point is $x = L/\sqrt{3}$. It remains to check the end points $x = 0$ and $x = c$, and compare these values depending on the ratio of c/L. This is left as an exercise to the reader. □

3.2. **Piano Competition.** 1000 people from all over the country are to gather and have a piano competition. 600 of the participants live in Seattle. They claim that to minimise total travel distance, the competition should be held in Seattle itself. The other 400 participants are furious and think they should calculate the average longitude and lattitude instead. Which side is correct?