MATH 31A DISCUSSION

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1. Differentiation

1.1. Basics. Given a function $f(x)$. The slope of the tangent line at $x = c$ is $f'(c)$.

1.1.1. *Higher Derivatives.* Recursively define higher derivatives: $f^{(n)} = (f^{(n-1)})'$.

1.1.2. *Chain Rule.* If f and g are differentiable, then $(f \circ g)(x) = f(g(x))$ is differentiable and $(f(g(x)))' = f'(g(x))g'(x)$.

1.2. Exercise 3.6.51. Show that a nonzero polynomial function $y = f(x)$ cannot satisfy the equation $y'' = -y$. Use this to prove that neither $\sin x$ nor $\cos x$ is a polynomial.

Proof. Let $y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, with $a_n \neq 0$. If the degree $n < 2$, then $y'' \equiv 0$ so $y'' \neq -y$. Otherwise, $y' = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1$, and $y'' = (n-1)na_n x^{n-2} + \ldots + 2a_2$. Since y'' lacks a monomial x^n, we cannot have $y'' = -y$. □

1.3. Exercise 3.7.92. A Discontinuous Derivative. Use the limit definition to show that $g'(0)$ exists but $g'(0) \neq \lim_{x\to 0} g'(x)$, where

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Proof. Recall $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$. So by definition, $g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \lim_{h\to 0} h \sin \frac{1}{h}$. Using Squeeze Theorem and $-|h| \leq h \sin \frac{1}{h} \leq |h|$, we get that $g'(0) = 0$. Away from $x = 0$, we can use the formula and get $g'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot (-1)\frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Now $\lim_{x\to 0} g'(x)$ does not exist since $\lim_{x\to 0} 2x \sin \frac{1}{x} = 0$ by Squeeze Theorem but $\lim_{x\to 0} \cos \frac{1}{x}$ does not exist. □

1.4. Exercise 3.8.35. Implicit Differentiation. If the derivative dx/dy exists at a point and $dx/dy = 0$, then the tangent line is vertical. Calculate dx/dy for the equation $y^4 + 1 = y^2 + x^2$ and find the points on the graph where the tangent line is vertical.

Solution. By implicit differentiation, we get $4y^3 = 2y + 2x\frac{dx}{dy}$. Setting $dx/dy = 0$, we get $4y^3 = 2y$, so $y = 0, \pm\frac{1}{\sqrt{2}}$. If $y = 0$ then $x = \pm 1$, if $y = \pm\frac{1}{\sqrt{2}}$, then $x^2 = \frac{3}{4}$, so we get 6 points. □

1.5. **Exercise 3.9.44. Related Rates.** A wheel of radius r is centred at the origin. As it rotates, the rod of length L attached at the point P drives a piston back and forth in a straight line. Let x be the distance from the origin to the point Q at the end of the rod.

(a) Use the Pythagorean Theorem to show that
$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

(b) Differentiate part (a) with resepct to t to prove that
$$2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0.$$

(c) Calculate the speed of the piston when $\theta = \frac{\pi}{2}$, assuming that $r = 10$ cm, $L = 30$ cm, and the wheel rotates at 4 revolutions per minute.

Solution. Parts (a) and (b) are straightforward. 4 revolutions per minute means $\frac{d\theta}{dt} = 4 \cdot 2\pi$ per minute. From part (a), we get $30^2 = x^2 + 10^2$, so $x = 20\sqrt{2}$. Plugging in, we get $2(20\sqrt{2} - 0)(\frac{dx}{dt} + 10 \cdot 8\pi) + 0 = 0$. So $\frac{dx}{dt} = -80\pi$ cm per minute. □

1.6. **Exercise 3.8.55. Lemniscate Curve.** The *lemniscate curve* $(x^2 + y^2)^2 = 4(x^2 - y^2)$ was discovered by Jacob Bernoulli in 1694, who noted that it is "shaped like a figure 8, or knot, or the bow of a ribbon." Find the coordinates of the four points at which the tangent line is horizontal.

Solution. By implicit differentiation and chain rule, we get $2(x^2 + y^2)(2x + 2yy') = 4(2x - 2yy')$. If $y' = 0$, we get $2(x^2 + y^2)(2x) = 4(2x)$, yielding $x^2 + y^2 = 2$. Substituting in to the lemniscate curve, we get $x^2 - y^2 = 1$. So $x^2 = 3/2$ and $y^2 = 1/2$. □