Math131A Midterm Solutions

Q1. Does the series $\sum \frac{(-1)^n n}{\sqrt{n!}}$ converge or diverge? Does it converge absolutely? Prove your claims.
[Recall that $n! = n(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1$.]

Solution. Let $s_n = \frac{n}{\sqrt{n!}}$. Note that for sufficiently large n, $n! > n(n-1)(n-2)(n-3)(n-4)(n-5) > (n-5)^6$, so $\frac{n}{\sqrt{n!}} < \frac{n}{(n-5)^3}$. By comparison test, the series $\sum s_n$ converges, so $\sum(-1)^n s_n$ converges absolutely, and hence converges. □

Q2. Prove that the inequality

$$\Big| |x| - |y| \Big| \leq |x-y|$$

holds for all real numbers $x, y \in \mathbb{R}$.

Solution. We need to show that $-|x-y| \leq |x| - |y| \leq |x-y|$. By the Triangle Inequality, we have $|x| = |x-y+y| \leq |x-y| + |y|$, yielding the second inequality $|x| - |y| \leq |x-y|$. By symmetry, exchanging the role of x and y, we get $|y| - |x| \leq |x-y|$, which gives the first inequality. □

Q3. Suppose (s_n) is a convergent sequence such that $\lim s_n < 23$. Prove that *eventually* $s_n < 23$; *i.e.*, there exists a number N such that $n > N$ implies $s_n < 23$.

Solution. Let $s = \lim s_n$. Let $\varepsilon = 23 - s$. As $s < 23$, $\varepsilon > 0$. Therefore by definition, there exists N such that $n > N$ implies $|s_n - s| < \varepsilon$. But then $s_n < s + \varepsilon = 23$, as desired. □

Q4. Suppose $A, B \subset \mathbb{R}$ are bounded nonempty subsets. Let $C = \{a - b : a \in A,\ b \in B\}$ be the set containing the <u>difference</u> $a - b$ for each $a \in A$ and $b \in B$. Calculate $\inf C$ in terms of $\inf A$, $\inf B$, $\sup A$, and $\sup B$. Prove your claim.

Solution. Let $\alpha = \inf A$ and $\beta = \sup B$. We prove that $\inf C = \alpha - \beta$.

Recall that $\inf C$ is the greatest lower bound of elements in C. As α is a lower bound of A, we have $\alpha \leq a$ for all $a \in A$. Also, as β is an upper bound of B, we get $b \leq \beta$ for all $b \in B$, and thus $-\beta \leq -b$ for all $b \in B$, giving that $\alpha - \beta \leq a - b$ for all $a \in A$ and $b \in B$. This establishes that $\alpha - \beta$ is a lower bound of C, *i.e.*, $\alpha - \beta \leq \inf C$.

Now we show that $\alpha - \beta$ is the greatest lower bound. Let $\varepsilon > 0$. It suffices to show that $\alpha - \beta + \varepsilon$ is not a lower bound of C. Indeed, consider $\alpha + \varepsilon/2$ and $\beta - \varepsilon/2$. Since $\alpha = \inf A$ is the greatest lower bound, $\alpha + \varepsilon/2$ is not a lower bound of A, and thus there exists $a \in A$ such that $a < \alpha + \varepsilon/2$. Similarly, as $\beta = \sup B$ is the least upper bound, $\beta - \varepsilon/2$ is not an upper bound of B, and thus there exists $b \in B$ such that $b > \beta - \varepsilon/2$. Adding the two inequalities, we get $a - b < \alpha - \beta + \varepsilon$. This means that $a - b \in C$ is a number smaller than $\alpha - \beta + \varepsilon$, which is therefore not a lower bound. □

Q5. Let (s_n) be a sequence of nonnegative numbers. Prove that $\sum s_n^p$ converges for <u>all</u> $p \geq 1$ if and only if it converges for $p = 1$.

Solution. The forward direction is obvious. Let us prove that if $\sum s_n$ converges, then so does $\sum s_n^p$ for $p \geq 1$. Suppose $\sum s_n$ is a convergent series, then $\lim s_n = 0$. Therefore there exists $N \in \mathbb{N}$ such that $n > N$ implies $s_n < 1$, which in turn gives $s_n^p < s_n$. By comparison, $\sum s_n^p$ converges.

Note that even though $s_n^p < s_n$ only for $n > N$, the comparison test still works. Indeed, the intial segment of a series does not change the convergence behaviour. If one wants to be careful, one may use comparison to conclude that $\sum_{n>N} s_n^p$ converges, and then state that $\sum s_n^p = \sum_{n \leq N} s_n^p + \sum_{n>N} s_n^p$, which is finite as both sums are. □

Q6. Let (a_n) be a sequence such that $\liminf |a_n| = 0$. Prove that it has a subsequence (b_n) such that both $\lim b_n$ and $\sum b_n$ converge.

Solution. As

$$\liminf |a_n| = \lim_k \inf\{|a_n| : n > k\} = 0,$$

for any $\varepsilon > 0$, there exists N such that $\inf\{|a_n| : n \geq N\} < \varepsilon$.

We construct a sequence (n_k) by induction. Consider the statement P_k which states that ther exists natural numbers $n_0 < n_1 < \ldots < n_k$ such that $|a_{n_i}| < 2^{-i}$. Obviously P_0 is true. Suppose that P_k is true. By definition, there exists $n_{k+1} > n_k$ such that $\inf\{|a_n| : n \geq n_{k+1}\} < 2^{-(k+1)}$. In particular, $|a_{n_{k+1}}| < 2^{-(k+1)}$. Therefore P_{k+1} holds and we finish the inductive construction of (n_k).

Since $|a_{n_k}| < 2^{-k}$, and $\sum 2^{-k} = 1$ is a convergent (geometric) series, $\sum |a_{n_k}|$ converges by comparison. This means $\sum a_{n_k}$ is absolutely convergent and thus is convergent. Furthermore, that implies $\lim a_{n_k}$ converges (to 0). □