My main research interests are computational complexity, discrete geometry, and combinatorics. I study tilings in this context, where tiles have simple geometries and the complexity arises from the combinatorial interactions of these simple tiles. Fix a lattice in Euclidean space and let Λ denote its cells. (Here we consider the triangular lattice in $\mathbb{R}^2$ and the cubic lattice in $\mathbb{R}^n$.) A **tile** is a finite subset of Λ. Let T be a set of tiles, called a **tileset**. A **tiling** π is a collection of translated[1] copies of tiles in T so that the tiles $\tau \in \pi$ are pairwise disjoint. A region $R \subset \Lambda$ is **tileable** (by T) if $R = \bigcup_{\tau\in\pi} \tau$ for some tiling π. We restrict attention to tiling finite regions with finite tilesets.

The main driving principle of my research program is that simple tiles can exhibit complex behaviors. I studied this phenomenon on the square lattice in the plane (Section 1)—which led to mentoring a related undergraduate research project (Section 2)—the cubic lattice in higher dimensions (Section 3), and the triangular lattice in the plane (Section 4). Some interdisciplinary projects in computational complexity are briefly mentioned in Section 5. Natural extensions for future projects are included towards the end of the appropriate sections, with a strong emphasis on problems suitable for undergraduate research.

1. Tiling simply connected regions with rectangles

The subject of tilings has been well studied [GS87], though classically the square lattice in the Euclidean plane was the focus (see *e.g.* [Gol65]). Even in this setting, deciding whether a region is tileable by a tileset is NP-complete [Lew78].[2] This is unsurprising given the power and richness of the tiling model. For example, tilings can perform arbitrary computations by emulating a universal Turing machine. Showing that a *simple* tileset could retain such computational power would be more challenging and interesting. In fact, the tileability problem remains NP-complete even when the tileset is fixed as the straight trominoes $\{[1\times 3],\ [3\times 1]\}$ [BNRR95]. However, when the region is simply connected, tileability by straight trominoes is in P [KK92]. By comparing these results, it is apparent that simple connectivity makes a difference. Using techniques in combinatorial group theory developed by Conway and Lagarias [CL90] and extending the height function approach [Thu90], Rémila showed that tileability of simply connected regions is in P for any two fixed rectangles [Rém05]. What about three or more rectangles? I constructed a tileset that, while consisting only of rectangular tiles, is still complicated computationally:

Theorem 1 ([PY13b]) *There is a finite set of rectangles such that tileability of simply connected regions is* NP*-complete.*

First we constructed a small tileset T such that tileability of simply connected regions by T is NP-complete. Then we proved a general lemma that transforms any given tileset T to a tileset T' consisting only of rectangles. It works by also providing a (linear time) algorithm for transforming a given simply connected region R to another simply connected region R' such that R is tileable by T if and only if R' is tileable by T'. This lemma applied to the tileset constructed above gives the desired set of rectangles.

Project 1 (Tiling by rectangles) Tileability of simply connected regions with 2 rectangles is in P; it is NP-complete with 117 rectangles [Yan13, §3]. Can this gap be tightened?

My first undergraduate research experience (via the Summer Undergraduate Research Fellowship program at Caltech) was working on tightening the gap between an upper and a lower bound [Yan10]. My mentor, Rick Wilson, explained to me that sometimes mathematics is furthered via small, incremental progress on difficult problems. Even if the ultimate goal to consider

[1] If rotations or reflections are desired, we may simply add rotated or reflected tiles into T.

[2] P is the class of decision problems solvable in polynomial time. NP is the class solvable in nondeterministic polynomial time. See *e.g.* [GJ79, Pap94] for definitions and background on computational complexity. We informally assume P $\neq$ NP and consider problems in P *easy* and NP-complete problems *hard*.

the behavior with 3 rectangles is beyond reach at the moment, lowering the number 117 is certainly a suitable project at the undergraduate level.

2. Counting tromino tilings (undergraduate research project)

Each NP decision problem has a natural **associated counting problem**. For example, instead of asking for the existence of tilings, one could ask for the *number* of such tilings. A region is tileable if the number of tilings is positive. As such, being able to count in polynomial time implies that the decision problem is in P. Conversely, it seems intuitive that an NP-complete problem would have a #P-complete counting version.[3] However, it is unknown whether this is always true. As such, a good source of reasonable projects is to take an NP-complete problem and prove that its counting version is #P-complete.[4]

Guided by this principle, I found a counting problem in tilings that is suitable for an undergraduate project. As mentioned above, tiling general regions with straight trominoes is NP-complete. Is the counting version #P-complete? I posed this problem to Kyle Meyer, then an undergraduate student spending a summer at the University of Minnesota participating in our Research Experiences for Undergraduates (REU) program. Besides explaining the relevant background and suggesting suitable reading material for him, I met with him periodically throughout the summer. He was indeed successful:

Project 2 (Counting tromino tilings) Together with REU student Kyle Meyer, we proved that it is #P-complete to count the number of tilings of general regions with straight trominoes [Mey14]. Is this also true for simply connected regions?

This follow-up project is more ambitious, since tiling simply connected regions by straight trominoes is in P, and so it is less clear how hard it is to count such tilings. It is not an unreasonable problem to consider, however, since there are tileability problems in P that are #P-complete to count. In particular, this can be seen in domino tilings in higher dimensions, which is discussed next.

3. Tiling by generalized dominoes in higher dimensions

Tiling by dominoes is equivalent to finding perfect matchings in the dual graph and so is in P (see *e.g.* [LP09]). Moreover, the number of matchings of a *planar* graph (and therefore the number of tilings) can be counted in polynomial time using a Pfaffian method [Kas67] (see also [Fis61, Kas61]). In higher dimensions, we consider juxtaposing two cubes together to get a $[2 \times 1 \times \cdots \times 1]$ domino and tiling with all its rotations. Tileability remains a matching problem and so is in P. However, using the fact that counting matchings in general graphs is hard [DL92], we proved the following:

Theorem 2 ([PY13a]) *Counting the number of tilings by a* $[2 \times 1 \times \cdots \times 1]$ *domino with rotations is* #P*-complete in three dimensions and higher.*

Moore [Moo] asked about the complexity of tiling with a $[2 \times 2 \times 1]$ slab, which we resolved:

Theorem 3 ([PY13a]) *Tileability by a* $[2 \times \cdots \times 2 \times 1 \times \cdots \times 1]$ *slab (that is not a domino) with rotations is* NP*-complete. Moreover, it is* #P*-complete to count the number of such tilings.*

We also proved that these results remain true even when restricted to tiling contractible[5] regions.

[3] #P is the class of counting problems associated to NP decision problems. See [Val79] for definitions and background. As above, we informally consider #P-complete problems *hard*.

[4] This is the expected result; to prove that it is not #P-complete would shock the world.

[5] Not only is a contractible region simply connected; all its homotopy groups are trivial.

Project 3 (Tiling by bricks) A natural extension is to generalize the results above to tiling by an $[n_1 \times n_2 \times \cdots \times n_d]$ brick (with rotations) for any $n_1, \ldots, n_d \in \mathbb{N}$.

4. Triangular puzzles

In this section, we tile on the triangular lattice in $\mathbb{R}^2$. Each cell is a unit equilateral triangle with three **edges**. Each edge on the boundary of tiles and regions is **labeled** by 0 or 1. Incident edges in a tiling must have equal labels. Consider the **puzzle pieces** τ_0, τ_1, and $\rho_{0,1}$ (see Figure 1), which are tiles that can be rotated. The number of tilings of triangular regions by puzzle pieces is

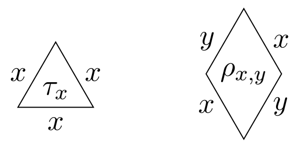

Figure 1. Tiles τ_x and $\rho_{x,y}$ with boundary edges labeled.

a Littlewood–Richardson coefficient [KTW04]. Therefore, tileability of triangular regions by puzzle pieces is in P [KT99]. On the other hand, we showed that tiling general regions is more complicated:

Theorem 4 ([PY16]) *Tileability of general regions by puzzle pieces is* NP*-complete.*

Project 4 (Counting general tilings by puzzle pieces) A natural follow-up question is whether counting the number of tilings of general regions by puzzle pieces is #P-complete. This is a reasonable project at the undergraduate level by the same guiding principle established in Section 2.

Littlewood–Richardson coefficients are ubiquitous. In particular, they appear as structure constants in the cohomology ring $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. In fact, the puzzle pieces were introduced to understand Schubert calculus, the study of intersection theory on a Grassmannian.

Project 5 (Puzzle complexity) Different puzzles are used in studying Schubert calculus. For example, in recent years, new tiles were introduced to study equivariant cohomology and K-theory of Grassmannians [KT03, Knu10], Gromov–Witten invariants [BKT03], GL_n Belkale–Kumar coefficients [KP11], and equivariant cohomology of two-step flag varieties [Buc15]. Each of these tilesets can be analyzed for their computational complexity.

5. Interdisciplinary projects

Computational complexity is relevant in other fields, which affords opportunities for interdisciplinary collaborations and explorations:

(1) Motivated by questions from biology, I proved that creating an optimal *antibiotic treatment plan* is NP-hard [TY15].
(2) I collaborated with electronics engineering researchers on *power management* problems based on combinatorial optimization [PYJL15].
(3) I currently have a project with Qie He, in the Department of Industrial and Systems Engineering at the University of Minnesota, regarding computational challenges in controlling *Markov processes.*

Viewing problems through the lens of computational complexity allows me to supervise and collaborate with undergraduate students with varied interests in adjoining fields.

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